138. Γ-Bundles and Almost Γ-Structures. II

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In [2], the author introduce the notion of almost Γ-structure and give an integrability condition of almost Γ-structures. It suggests us that there seems to be useful that to use some differential geometric aspects of tangent microbundles in the study of Γ-structures. In this note, we treat pseudoconnections of topological microbundles which was defined in [2], and show the following theorem: There is a 1 to 1 correspondence between the set of equivalence classes of Γ-structures on X and the set of Γ-equivalence classes of pseudoflat Γ-bundle structures of the diagram $X \xrightarrow{\Delta} X \times X \xrightarrow{p} X$, where Δ is the diagonal map and p is the projection to the first component (cf. [4]). Notations of this note are similar that of [1], [2].

1. *Pseudoconnection of topological microbundles.* For an element a of $\boldsymbol{R}^n$, we define the parallel transformation t_a by

$$t_a(b)=b-a,\ b \in \boldsymbol{R}^n.$$

Lemma 1. *A homeomorphism f from a neighborhood of the origin of $\boldsymbol{R}^n$ into $\boldsymbol{R}^n$ is a parallel transformation if and only if*

$$(df)(a, b)=b-a,$$

where $(df)(a, b)=f(b)-f(a)$.

On the other hand, if $\alpha_U \in C^1(U, \boldsymbol{R}^n)$, then

$$(1)\qquad \delta(t_{\alpha_U})(x_0, x_1, x_2)=t_{(\delta\alpha_U)(x_0,x_1,x_2)},$$

where the multiplication is defined to be the compositions of $\boldsymbol{R}^n$ and $\delta(t_{\alpha_U})$ and $(\delta\alpha_U)$ are given by

$$\delta(t_{\alpha_U})(x_0, x_1, x_2)=t_{\alpha_U(x_1,x_2)}t_{\alpha_U(x_0,x_2)}^{-1}t_{\alpha_U(x_0,x_1)},$$

$$(\delta\alpha_U)(x_0, x_1, x_2)=\alpha_U(x_1, x_2)-\alpha_U(x_0, x_2)+\alpha_U(x_0, x_1).$$

Definition. Let $\{\varphi_{UV}(x)\}$ be a transition function of an n-dimensional topological microbundle ξ over normal paracompact topological space X, then a collection $\{t_{\alpha_U(x,y)}\}$, $\alpha_U(x, y) \in C^1(U, \boldsymbol{R}^n)$, is called a pseudoconnection of $\{\varphi_{UV}(x)\}$ if $\{t_{\alpha_U}\}$ satisfies

$$(2)\qquad \varphi_{UV}(x)^{-1}t_{\alpha_U(x,y)}\varphi_{UV}(y)=t_{\alpha_V(x,y)}.$$

According to [1], we call the collection $\{\delta(t_{\alpha_U})\}$ to be the curvature form of $\{t_{\alpha_U}\}$.

Definition. We call $\{t_{\alpha_U}\}$ is a flat pseudoconnection if the curvature form $\{\delta(t_{\alpha_U})\}$ of $\{t_{\alpha_U}\}$ is equal to 0.

Definition. $\{\varphi_{UV}(x)\}$ is called a pseudoflat microbundle if $\{\varphi_{UV}(x)\}$ has a flat pseudoconnection.

Example. Let $X=\{U, h_U\}$ is a paracompact topological manifold, then setting

$$h_{U,x}(y)=h_U(y)-h_U(x),\ g_{UV}(x)=h_{U,x}h_{V,y}^{-1},$$

$\{g_{UV}(x)\}$ is a pseudoflat microbundle with flat pseudoconnection $\{s_{U,x,y}\}$, where $s_{U,x,y}$ is given by

$$s_{U,x,y}=h_{U,x}h_{U,y}^{-1}.$$

Lemma 2. *If $\{t_{\alpha_U}\}$ is a flat pseudoconnection, then with suitable covering system of X, we can write*

$$t_{\alpha_U(x,y)}=t_{\beta_U(y)}t_{\beta_U(x)}^{-1}. \tag{3}$$

Definition. A flat pseudoconnection is called non-degenerate if each β_U given by (3) defines a homeomorphism from U into $\boldsymbol{R}^n$.

Note. This definition does not depend on the choice of $\{\beta_U\}$.

Since h_U is a homeomorphism from U into $\boldsymbol{R}^n$, the pseudoconnection $\{s_{U,x,y}\}$ of $\{g_{UV}(x)\}$ given in the above example is non-degenerate.

Lemma 3. *If X is a paracompact n-dimensional topological manifold, $\{\varphi_{UV}(x)\}$ is a transition function of an n-dimensional topological microbundle ξ and $\{\varphi_{UV}(x)\}$ has a non-degenerate flat pseudoconnection, then ξ is the tangent microbundle of x.*

2. *Transition function comes from the structure.* In the rest, we assume that X is a paracompact manifold and fix its dimension n.

Definition. A transition function $\{g'_{UV}(x)\}$ of the tangent microbundle of X is called a transition function comes from the (topological) structure of X if we can set

$$g'_{UV}(x)=h'_{U,x}h'_{V,x}{}^{-1},\ h'_{U,x}(y)=h'_U(y)-h'_U(x),$$

where h'_U is a homeomorphism from U into $\boldsymbol{R}^n$.

Definition. If in the above, $\{U, h'_U\}$ defines a Γ-structure of X, then we call $\{g'_{UV}(x)\}$ comes from the Γ-structure of X.

In the rest, we assume that the elements of Γ are the homeomorphisms from some open set of $\boldsymbol{R}^n$ into $\boldsymbol{R}^n$. Here n is the (fixed) dimension of X.

Lemma 4. *A transition function $\{\varphi_{UV}(x)\}$ of the tangent microbundle of X comes from the structure of X if and only if $\{\varphi_{UV}(x)\}$ has a non-degenerate flat pseudoconnection.*

We fix a transition function $\{g_{UV}(x)\}=\{h_{U,x}h_{V,x}^{-1}\}$ of the tangent microbundle τ of X and its non-degenerate flat pseudoconnection $\{s_{U,x,y}\}=\{h_{U,x}h_{U,y}^{-1}\}$. Here $\{U, h_U\}$ defines the topological structure of X. Then another transition function of τ is written as $\{\varphi_U(x)g_{UV}(x)\varphi_V(x)^{-1}\}$, where $\varphi_U(x)$ is a homeomorphism from some open set of $\boldsymbol{R}^n$ into $\boldsymbol{R}^n$ and $\varphi_U(x)(0)=0$ for any x and U. For this $\{\varphi_U(x)\}$, we obtain by lemma 1 and lemma 4,

Theorem 1. $\{\varphi_U(x)g_{UV}(x)\varphi_V(x)^{-1}\}$ *comes from the structure of*

X if and only if $\varphi_U(x)$ *satisfies*

(4) $$d(\varphi_U(x)s_{U,x,y}\varphi_U(y)^{-1})(a, b)=b-a,$$

for all U, with suitable covering system of X.

By lemma 1, we can rewrite (4) as

(4)′ $$\varphi_U(x)s_{U,x,y}\varphi_U(y)^{-1}=t_{\alpha_U(x,y)},\quad \alpha_U\in C^1(U, \boldsymbol{R}^n).$$

For this α_U, we define the map $\alpha_U{}^{\#}: U\times U\to U\times \boldsymbol{R}^n$ by

(5) $$\alpha_U{}^{\#}(x, y)=(x, \alpha_U(x, y)).$$

Then we can give a microbundle structure of the diagram $X\xrightarrow{\Delta} X\times X\xrightarrow{p} X$ by the diagram

$$\begin{array}{ccccc} & & U\times U & & \\ & {}^{\Delta}\nearrow & & \searrow^{p} & \\ U & & \Big\downarrow{\scriptstyle \alpha_U{}^{\#}} & & U \\ & {}_{\times 0}\searrow & & \nearrow_{p} & \\ & & U\times \boldsymbol{R}^n & & \end{array},$$

Note. The microbundle structure of the diagram $X\xrightarrow{\Delta} X\times X\xrightarrow{p} X$ given by this way, is pseudoflat and the collection $\{t_{\alpha_U}\}$ becomes its (non-degenerate flat) pseudoconnection.

3. *Classification of Γ-structures.* As in [2], we denote by Γ a pseudogroup consisted by a set of homeomorphisms from some open sets of $\boldsymbol{R}^n$ into $\boldsymbol{R}^n$. We assume that Γ contains all parallel transformations of $\boldsymbol{R}^n$, where n is the fixed dimension of X.

In [2], we define an almost Γ-structure on X as the collection $\{\varphi_U(x)\}$ such that

(6) $$\varphi_U(x)g_{UV}(x)\varphi_V(x)^{-1}\in \Gamma_{*x},$$

where $\varphi_U(x)$ is a cross-section from U into $H_*(n)_c$. If $\{U'\}$ is a refinement of $\{U\}$, then we identify the almost Γ-structure $\{\varphi_U(x)\}$ and $\{\varphi_U(x)\mid U'\}$.

Lemma 5. *There is a refinement $\{U'\}$ of $\{U\}$ and a representation $\{\varphi_{U'}(x)\}$ of $\{\varphi_U(x)\mid U'\}$ such that*

(6)′ $$\varphi_{U'}(x)g_{U'V'}(x)\varphi_{V'}(x)^{-1}\in \Gamma.$$

For the simplicity, we denote $\{U\}$ and $\{\varphi_U(x)\}$ instead of $\{U'\}$ and $\{\varphi_{U'}(x)\}$.

Since Γ contains all parallel transformations of $\boldsymbol{R}^n$, if the above $\{\varphi_U(x)\}$ satisfies (4), then we get

(7) $$\varphi_U(x)s_{U,x,y}\varphi_U(y)^{-1}\in \Gamma,$$

which is the integrability condition of almost Γ-structures (cf. [2], (11)).

Lemma 6. *If the collection $\{\varphi_U(x)\}$ and $\{\psi_U(x)\}$ both satisfy* (4), (6)′ *and the following*

(8)′ $$\varphi_U(x)\psi_U(x)^{-1}\in \Gamma,$$

then they define same Γ-structure of X.

Proof. We set

$$\varphi_U(x)s_{U,x,y}\varphi_U(y)^{-1}=t_{\alpha_U(x,y)},$$
$$\psi_U(x)s_{U,x,y}\psi_U(y)^{-1}=t_{\alpha_U'(x,y)},$$

then since $\{t_{\alpha_U}\}$ and $\{t_{\alpha_U'}\}$ are flat pseudoconnections, we can set

$$\alpha_U(x, y)=\beta_U(y)-\beta_U(x),\ \alpha_U'(x, y)=\beta_U'(y)-\beta_U'(x),$$

by using suitable cobering system of X, which is also denoted by $\{U\}$. Since we obtain

$$t_{\beta_U(x)}{}^{-1}\varphi_U(x)h_{U,x}=t_{\beta_U(y)}{}^{-1}\varphi_U(y)h_{U,y},$$
$$t_{\beta_U'(x)}{}^{-1}\psi_U(x)h_{U,x}=t_{\beta_U'(y)}{}^{-1}\psi_U(y)h_{U,y},$$

we may denote

$$h_U{}^1=t_{\beta_U}{}^{-1}{}_{(x)}\varphi_U(x)h_{U,x},$$
$$h_U{}^2=t_{\beta_U'(x)}{}^{-1}\psi_U(x)h_{U,x}.$$

Then by (6)′, using suitable covering system $\{U, h_U{}^1\}$, and $\{U, h_U{}^2\}$ define Γ-structures on X and since we have by (8)

$$h_U{}^1(h_U{}^2)^{-1}=t_{\beta_U(x)}{}^{-1}\varphi_U(x)\psi_U(x)^{-1}t_{\beta_U'(x)}\in\Gamma,$$

the Γ-structures defined by them are equivalent. Therefore we get the lemma.

Since a Γ-structure on X is defined by a collection $\{U, h_U'\}$, $h_U'h_V'^{-1}\in\Gamma$, any Γ-structure on X has a transition function comes from the Γ-structure. Therefore we obtain by theorem 1 and lemma 6,

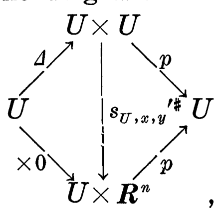
Theorem 2. *Two Γ-structures on X are equivalent if and only if $\{\varphi_U(x)\}$ and $\{\psi_U(x)\}$ satisfy* (8) *by suitable covering system of X.*

Here $\{\varphi_U(x)g_{UV}(x)\varphi_V(x)^{-1}\}$ and $\{\psi_U(x)g_{UV}(x)\psi_V(x)^{-1}\}$ are the transition functions comes from these two Γ-structures.

Note. If Γ_1 is a subpseudogroup of Γ and contains all parallel transformations of R^n, then we get similar theorem about the classification of Γ_1-structures on a Γ-manifold.

4. *Γ-equivalence of pseudoflat Γ-bundle structures of the diagram $X\xrightarrow{\Delta}X\times X\xrightarrow{p}X$.*

Definition. Let $\{U, h_U'\}$ be a Γ-structure on X, then the microbundle structure of the diagram $\xrightarrow{\Delta}X\times X\xrightarrow{p}X$ given by



,

$s_{U,x,y}'=h_{U,x}'h_{U,y}'^{-1}$, is called the associated (pseudoflat) Γ-bundle of the Γ-structure $\{U, h_U'\}$.

Since a microbundle structure of the diagram $X\xrightarrow{\Delta}X\times X\xrightarrow{p}X$ is given by the diagram

$$
\begin{array}{ccccc}
 & & U\times U & & \\
 & {\scriptstyle\Delta}\nearrow & \Big| & \searrow{\scriptstyle p} & \\
U & & \Big\downarrow{\scriptstyle \alpha_U{}^{\#}} & & U \\
 & {\scriptstyle\times 0}\searrow & & \nearrow{\scriptstyle p} & \\
 & & U\times \boldsymbol{R}^n & &
\end{array},
$$

we denote this microbundle structure by $\mathfrak{X}(\{\alpha_U\})$. Then we have

Lemma 7. *$\mathfrak{X}(\{\alpha_U\})$ is a pseudoflat microbundle if and only if*

$$(\delta\alpha_U)(x, y)=0. \tag{9}$$

If $\mathfrak{X}(\{\alpha_U\})$ is a pseudoflat microbundle, then we can set

$$\alpha_U(x, y)=\beta_U(y)-\beta_U(x),$$

and the transition function of $\mathfrak{X}(\{\alpha_U\})$ is given by $\{\beta_{U,x}\beta_{V,x}{}^{-1}\}$. Therefore we get

Lemma 8. *$\mathfrak{X}(\{\alpha_U\})$ is a Γ-bundle if and only if the germ of $\beta_{U,x}\beta_{V,x}{}^{-1}$ at x belongs in Γ_{*x} for all $x\in X$ and $\{U, V\}$.*

If $\mathfrak{X}(\{\alpha_U\})$ and $\mathfrak{X}(\{\alpha_U'\})$ are two microbundle structures of the diagram $X\xrightarrow{\Delta}X\times X\xrightarrow{p}X$, then there is a collection of homeomorphisms f_U from $U\times\boldsymbol{R}^n$ to $U\times\boldsymbol{R}^n$ such that the diagram

$$
\begin{array}{ccc}
 & & U\times \boldsymbol{R}^n \\
 & {\scriptstyle \alpha_U{}^{\#}}\nearrow & \\
U\times U & & \Big\downarrow{\scriptstyle f_U} \\
 & {\scriptstyle \alpha_U{}'^{\#}}\searrow & \\
 & & U\times \boldsymbol{R}^n
\end{array}, \tag{10}
$$

is commutative for any U. We denote the germ of the map $f_{U,x}\colon \boldsymbol{R}^n\longrightarrow\boldsymbol{R}^n$ defined by $f_{U,x}(a)=f_U(x, a)$ in $H_*(n)_x$ by $\bar{f}_{U,x}$.

Definition. Two (pseudoflat) Γ-bundle structures $\mathfrak{X}(\{\alpha_U\})$ and $\mathfrak{X}(\{\alpha_U'\})$ of the diagram $X\xrightarrow{\Delta}X\times X\xrightarrow{p}X$ is called Γ-equivalent if $\bar{f}_{U,x}$ defined by the diagram (10) belongs in Γ_{*x} for any x and U.

Using this terminology, theorem 2 is rewritten as follows.

Theorem 2′. *There is a 1 to 1 correspondence between the set of equivalence classes of Γ-structures on X and the set of Γ-equivalence classes of pseudoflat Γ-bundle structures of the diagram $X\xrightarrow{\Delta}X\times X\xrightarrow{p}X$.*

Note. We can obtain similar theorem concerning the classification of Γ_1-structures on a Γ-manifold.

5. *A uniqueness condition of Γ_1-structures of Γ-manifolds.* We assume Γ_1 is a subpseudogroup of Γ which contains all parallel transformations of $\boldsymbol{R}^n$. The map from $H^1(x, \Gamma_{1*c})$ into $H^1(x, \Gamma_{1*c})$ induced from the inclusion of Γ_1 into Γ is denoted by τ^*. We also assume that the pair (Γ, Γ_1) satisfies following condition.

(U). If we fix a Γ-structure of $\boldsymbol{R}^n$, then this Γ-manifold $\boldsymbol{R}^n$ has at most one Γ_1-structure.

Lemma 9. *If $\varphi_U(x)$ satisfies (4), then we have*

$$\varphi_U(y)(a)=\varphi_U(x)(a+h_U(y)-h_U(x)) \\ -\varphi_U(x)(h_U(y)-h_U(x)),\ x, y\in U,\ a\in \boldsymbol{R}^n. \tag{11}$$

Corollary. *If $\varphi_U(x)$ satisfies (4), then its value on U is determined uniquely by its value at a point of U.*

Note. By (11), if the values of φ_U are diffeomorphisms, then we get

$$J_0(\varphi_U(y))=J(\varphi_U(x))(h_U(y)-h_U(x)). \tag{12}$$

Using theorem 2 and lemma 9, we can prove

Theorem 3. *If the pair (Γ, Γ_1) satisfies the condition (U) and the map $\tau^*: H^1(X, \Gamma_{1*c})\longrightarrow H^1(X, \Gamma_{*c})$ is injective for all paracompact manifold X, then a Γ-manifold has at most one Γ_1-structure.*

Corollary. *(Cf. [2], [3], [5], [6]). If Γ_1-structure of $\boldsymbol{R}^n$ is unique and the homotopy types of Γ_0 and $\Gamma_{1,0}$ (cf. [2], n°4) are same, then a Γ_1-manifold has at most one Γ_1-structure. Here the topology of $\Gamma_{1,0}$ is that of induced from Γ_0, but they need not be the compact open topology.*

Note. Denoting $\mathfrak{R}^n$ the group of all parallel transformations of $\boldsymbol{R}^n$, then $\mathfrak{R}^n$ is contained in any Γ_1 and $\boldsymbol{R}^n$ has an $\mathfrak{R}^n$-structure. Therefore, for any Γ_1, $\boldsymbol{R}^n$ becomes a Γ_1-manifold. But the Γ_1-structures of $\boldsymbol{R}^n$ are not unique in general. For example, the complex structures of $\boldsymbol{R}^{2n}$ are not unique although $n=1$.

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