132. A Fixed Point Theorem for Contraction Mappings in a Uniformly Convex Normed Space

By Shouro KASAHARA
Kobe University

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The purpose of this note is to prove the following

Theorem 1. *Let A be a nonempty, weakly compact and convex subset of a uniformly convex normed space,*[1] *and $\mathcal{F}$ be a nonempty commutative family of contraction mappings*[2] *of A into itself. Then the set of all common fixed points for $\mathcal{F}$ is nonempty, closed and convex.*

This follows from Theorems 2 and 3 below.

Following Brodskii and Milman [1], we say that a bounded convex subset S of a normed space has *normal structure* provided for each convex subset B of S which contains more than one point, there exists a point $a \in B$ such that $\sup_{y \in B} \| a-y \| < d(B)$, where $d(B)$ denotes the diameter of B. A point $a \in B$ is said to be a *diametral point* of B if $\sup_{y \in B} \| a-y \| = d(B)$.

Theorem 2. *Each bounded convex subset of a uniformly convex normed space has normal structure.*

Proof. It is easily seen that in a normed space E if a bounded subset $B \subset E$ which contains more than one point has a nondiametral point $a \in B$, then λa is a nondiametral point of λB for every $\lambda \neq 0$, and $x+a$ is a nondiametral point of $x+B$ for every $x \in E$. Therefore it is sufficient to show that in a uniformly convex normed space, each bounded convex subset B of diameter 1 which has $\{0\}$ as a proper subset, contains a nondiametral point of it.

Assume that 0 is a diametral point of B. Then we can find a sequence $\{a_n\}_{n \geq 2}$ of points of B such that

$$1 \geq \| a_n \| > 1 - \frac{1}{n} \qquad \text{for every } n \geq 2.$$

Suppose that the sequence $\{(1/2)a_n\}_{n \geq 2}$ consists of diametral points of B. Then there exists a sequence $\{b_n\}_{n \geq 2}$ of points of B such that

$$1 \geq \left\| b_n - \frac{1}{2} a_n \right\| > 1 - \frac{1}{n} \qquad \text{for every } n \geq 2,$$

1) A normed space is said to be *uniformly convex* if $\| x_n \| \leq 1$, $\| y_n \| \leq 1$, and $\lim \| x_n + y_n \| = 2$ imply $\lim \| x_n - y_n \| = 0$.

2) A mapping f of a subset A of a normed space into A is called a *contraction mapping* if $\| f(x) - f(y) \| \leq \| x-y \|$ for all $x, y \in A$.

and hence $\lim_{n\to\infty} \| b_n+(b_n-a_n) \|=2$. Now since $\| b_n \|\leq 1$ for every $n\geq 2$, and since $\lim_{n\to\infty} \| b_n-(b_n-a_n) \|=1$, the uniform convexity of the space implies that there exists an $n\geq 2$ such that $\| b_n-a_n \|>1$, which is absurd. Consequently, the sequence $\{(1/2)a_n\}_{n\geq 2}$ contains a nondiametral point of B.

Recently, Kirk [2] has proved the following

Theorem. *Let A be a nonempty, bounded, closed, and convex subset of a reflexive Banach space, and suppose that A has normal structure. Then each contraction mapping of A into itself has a fixed point.*

In this theorem, the assumption reflexivity of the space can be weakened, and in fact we can show the validity of the following theorem by the method of Kirk. However, for the sake of completeness, we give the proof.

Theorem 3. *Let A be a nonempty, weakly compact and convex subset of a normed space, and suppose that A has normal structure. Then each contraction mapping f of A into itself has a fixed point.*

Proof. Since A is weakly compact, we can show by using Zorn's lemma that there exists a minimal nonempty weakly compact and convex subset $B\subset A$ which is invariant under f. Let

$$r_x(B)=\sup_{y\in B} \| x-y \|, \qquad r(B)=\inf_{x\in B} r_x(B),$$

and let, for each positive integer n,

$$B_n(x)=B\cap\left[x+\left(r(B)+\frac{1}{n}\right)V\right],$$

where V denotes the closed unit ball of the space. It is easy to see that the sets $C_n=\bigcap_{x\in B} B_n(x)$ form a decreasing sequence of nonempty weakly closed and convex subsets of the weakly compact set B, and hence $B_0=\bigcap_{n=1}^{\infty} C_n=\{x\in B;\, r_x(B)=r(B)\}$ is nonempty weakly compact and convex subset of B. We shall show that $B_0=B$. To this end, it suffices, by the minimality of B, to prove that B_0 is invariant under f. Let $x\in B_0$. Then for every $y\in B$, we have $\| f(x)-f(y) \|\leq \| x-y \|\leq r(B)$, and so $f(B)$ is contained in the set $f(x)+r(B)V$. Therefore the nonempty weakly compact and convex set $B\cap[f(x)+r(B)V]$ is invariant under f, and hence the minimality of B implies that B is contained in the set $f(x)+r(B)V$, which shows that $f(x)\in B_0$. Thus B_0 is invariant under f, and so $B_0=B$. Now suppose that the set B contains more than one point. Then, since A has normal structure, we can find a nondiametral point $a\in B$. Hence $r_a(B)<d(B)$. If $x, y\in B_0$, then $\| x-y \|\leq r_x(B)=r(B)$, and so we have

$$d(B_0)=\sup_{x,y\in B_0} \| x-y \|\leq r(B)\leq r_a(B)<d(B),$$

which contradicts the fact that $B_0=B$. Thus the set B consists of a single point, and the proof is completed.

We shall now proceed to prove Theorem 1.

For each $f\in\mathcal{F}$, we denote by F_f the set of all fixed points of f. From Theorems 2 and 3, it follows at once that each F_f is nonempty. We shall show that each F_f is convex. Let $x, y\in F_f$, $0\leq\lambda\leq 1$, and let $z=\lambda(x-y)+y=(1-\lambda)(y-x)+x$. Then we have

$$\|x-y\|\leq\|x-f(z)\|+\|y-f(z)\|\leq\|x-z\|+\|y-z\|=\|x-y\|,$$

and hence we have $\|x-f(z)\|=\|x-z\|$ and $\|y-f(z)\|=\|y-z\|$. On the other hand, since a uniformly convex normed space is strictly convex,[3] $f(z)\neq z$ implies

$$\left\|x-\frac{1}{2}(z+f(z))\right\|<\|x-z\| \text{ and } \left\|y-\frac{1}{2}(z+f(z))\right\|<\|y-z\|,$$

which yield a contradiction $\|x-y\|<\|x-y\|$. Consequently $z\in F_f$, and F_f is convex.

As can readily be seen, F_f is closed, and hence it is weakly closed. Consequently for every $f_1, f_2, \cdots, f_n\in\mathcal{F}$, the set $F_n=\bigcap_{i=1}^{n} F_{f_i}$ is weakly compact and convex. In addition, if $f\in\mathcal{F}$, then for every $x\in F_n$, we have

$$f(x)=ff_i(x)=f_if(x) \qquad \text{for each } i=1, 2, \cdots, n,$$

and hence $f(x)\in F_n$. Thus F_n is invariant under each $f\in\mathcal{F}$. Therefore if F_n is nonempty, then by Theorems 2 and 3, each $f\in\mathcal{F}$ has a fixed point in F_n, that is $F_f\cap F_n\neq\varnothing$. This shows that the family $\{F_f; f\in\mathcal{F}\}$ of weakly closed and convex subsets of the weakly compact set A has the finite intersection property. Hence the family has a nonempty intersection, which is closed and convex.

References

[1] M. S. Brodskii and D. P. Milman: On the center of a convex set. (Russian) Dokl. Akad. Nauk SSSR (N.S.), **59**, 837-840 (1948).

[2] W. A. Kirk: A fixed point theorem for mappings which do not increase distances. Amer. Math. Monthly, **72**, 1004-1006 (1965).

3) A normed space is *strictly convex* if $\|x+y\|<2$ whenever $\|x\|=\|y\|=1$ and $x\neq y$.