121. On the Separation Theorem of Intermediate Propositional Calculi

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In this paper is shown a sufficient condition for intermediate propositional calculi to have the separation theorem on them. By an intermediate propositional calculus we mean a calculus between the classical and the intuitionistic obtained by adding some new axioms to an intuitionistic propositional calculus. And by the separation theorem we mean the following

Theorem. *A provable formula in the calculus can be proved using only the axioms for implication and those for the logical symbols actually appearing in the formula.*

This theorem depends upon the axiom system of the calculus. And we call the calculus and its axiom system as separable if the separation theorem holds on the calculus. An example of separable intuitionistic systems is given in [3] and those of separable classical systems are in [4] and [5].

A formula is called an I (or C, or D, or N) formula if it contains only implication (or conjunction, or disjunction, or negation) as its logical symbols. An IC formula is a formula in which no logical symbols other than implication and conjunction are contained. An IC axiom is an axiom which is an IC formula. An IC theorem is a theorem which is an IC formula and is provable from IC axioms. An IC proof is a proof in which only IC axioms are used. A calculus is IC complete if the theorems which are IC formulas are IC theorems. And other combinations are defined similarly.

What is proved in this paper is that *if an intermediate propositional calculus satisfies the following* (1), (2), *and* (3), *it is separable.*

(1) *The axiom system of the calculus is constructed by adding some new* I *axioms to the axiom system of a separable intuitionistic propositional calculus. And the rule of substitution is in it.*

(2) *The calculus is* I *complete.*

(3) *There exist* I *formulas* $f_i(a, b)$ $(i=1, \cdots, n)$ *whose propositional variables are only* a *and* b *such that formulas of the forms*

$$a \vee b \supset f_i(a, b) \qquad (i=1, \cdots, n)$$

$$f_1(a, b) \supset \cdots \supset f_n(a, b) \supset a \vee b$$

are ID *theorems*. (We associate to the right, and $\supset$ binds less strongly than other logical symbols.)

Of these three, (1) and (2) will be accepted quite naturally, but (3) seems to be rather strong. And there arises a natural question if (1) and (2) would give a neccessary and sufficient condition. But it is yet open.

The condition (2) might seem to be contained in (1), but the fact is not. A counterexample is seen in [6].

Now we give the proof that a calculus satisfying above (1), (2), and (3) is separable. For that purpose, we prove that each of its fragments concerning the logical symbols is complete.

Lemma 1. *The calculus is* I (*or* ICDN) *complete.*

Proof. By (2), it is I complete. Since all the theorems are ICDN theorems, the calculus is ICDN complete.

Lemma 2. *The calculus is* IC *complete.*

Proof. Suppose that A is a provable IC formula. We transform this A as follows.

(i) *A subformula of the form* $B \& C \supset C$ *is replaced by* $B \supset C \supset D$.

(ii) *A subformula of the form* $B \supset C \& D$ *is replaced by* $(B \supset C) \& (B \supset D)$.

We call this transformation as C transformation. By this transformation, A will be transformed into a formula of the form $B_1 \& \cdots \& B_m$ where B_i $(i=1, \cdots, m)$ is an I formula. Since $\vdash (B \& C \supset D) \equiv (B \supset C \supset D)$ and $\vdash (B \supset C \& D) \equiv (B \supset C) \& (B \supset D)$ intuitionistically, $B_1 \& \cdots \& B_m$ is provable in the calculus. And also $\vdash B_i (i=1, \cdots, m)$ in the calculus. And there is an I proof for each B_i since B_i is a theorem and an I formula. On the other hand, $B_1 \supset \cdots \supset B_m \supset B_1 \& \cdots \& B_m$ is an intuitionistic IC theorem, so $B_1 \& \cdots \& B_m$ is an IC theorem. But $B_1 \& \cdots \& B_m \supset A$ is an intuitionistic IC theorem, so A is an IC theorem.

Lemma 3. *The calculus is* IN *complete.*

Proof. Suppose that A is a provable IN formula. We transform A as follows.

(iii) *A subformula of the form* $\neg B$ *is replaced by* $B \supset c$, *where* c *is a propositional variable which does not appear in* A. (We put the so changed formula as A*. And negation does not appear in A^*.)

(iv) *We transform* A^* *into* $(c \supset B_1) \supset \cdots \supset (c \supset B_m) \supset A^*$, *where* $B_1, \cdots, B_m$ *are all the subformulas of* A^*.

We call this transformation as N transformation. We put the transformed formula as A^{**}. Since A is provable, A^{**} is provable in the calculus. And since A** is an I formula, there is an I proof for A^{**}. We replace c by $\neg(c \supset c)$ in A^{**} and in its proof, and call the obtained formula as A^{***}. This A^{***} is provable from I axioms. Since $\vdash A^{***} \equiv A$ intuitionistically, there is an intuitionistic IN proof for $A^{***} \supset A$. So there is an IN proof for A.

Lemma 4. *The calculus is* ID *complete.*

Proof. Suppose that A is a provable ID formula. We transform A as follows.

(v) *A subformula of the form* $B \vee C$ *is replaced by* $f_1(B, C) \& \cdots \& f_n(B, C)$.

We put the transformed formula as A^*. By (3), $\vdash A^* \equiv A$ in the calculus. We transform A^* by C transformation and get $D_1 \& \cdots \& D_k$. Since A is provable, $D_1, \cdots, D_k$ are provable. Further they are I theorems. By (3),

$$(B_1 \vee C_1 \supset f_1(B_1, C_1)) \supset \cdots \supset (B_1 \vee C_1 \supset f_n(B_1, C_1)) \supset (f_1(B_1, C_1) \supset \cdots \supset f_n(B_1, C_1) \supset B_1 \vee C_1) \supset \cdots \supset (B_m \vee C_m \supset f_1(B_m, C_m)) \supset \cdots \supset (B_m \vee C_m \supset f_n(B_m, C_m)) \supset (f_1(B_m, C_m) \supset \cdots \supset f_n(B_m, C_m) \supset B_m \vee C_m) \supset D_1 \supset \cdots \supset D_k \supset A$$

is an intuitionistic ID theorem, where $B_1 \vee C_1, \cdots, B_m \vee C_m$ are the subformulas replaced in the transformation (v). But all the antecedents of $D_1 \supset \cdots \supset D_k \supset A$ are ID theorems by (3), and $D_1, \cdots, D_k$ are I theorems, so A is an ID theorem.

Lemma 5. *The calculus is* ICN (*or* IDN) *complete.*

Proof. For a provable ICN (or IDN) formula, we do the N transformation. Then there is an IC (or ID) proof for it. And as in the proof of Lemma 3, we can obtain an ICN (or IDN) proof for it.

Lemma 6. *The calculus is* ICD *complete.*

Proof. This is obvious from Lemma 4 and Lemma 2.

By the above six lemmas we know that the calculus satisfying (1), (2), and (3) is separable. (It will be easily seen that we need not show the completeness concerning other combinations as CN, etc.)

As an example of separable intermediate propositional calculi, we show the following

Corollary. *The intermediate calculus obtained from an intuitionistic propositional calculus by adding a new axiom*

$$((a \supset b) \supset c) \supset ((b \supset a) \supset c) \supset c$$

is separable.

This calculus was studied by Dummett [2] and named LC. And the above axiomatization is due to Bull [1]. This obviously satisfies (1). The I completeness proof is given in [1]. Further,

$a\vee b\supset((a\supset b)\supset b)$, $a\vee b\supset((b\supset a)\supset a)$, and $((a\supset b)\supset b)\supset((b\supset a)\supset a)\supset a\vee b$ are ID theorems (cf. [2]). So (3) is satisfied.

The algebraic proof of this corollary will appear in near future.

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