159. An Extension of a Generalized Measure

By Masahiro TAKAHASHI

Department of Mathematics, Nara Medical College

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1. Introduction. The notion of a topological-additive-group-valued measure has been introduced by the author in [1], which states the process of extending such a measure defined for a certain class of simple sets to one for a wider class of sets. It is noted that any set for which the extended measure is defined is necessarily contained in some set for which the original measure is defined. When, for example, the original measure is given for a class of finite sums of half open intervals in the real line, a measurable set with respect to the extended measure is always bounded.

The purpose of this paper is to extend a topological-additive-group-valued measure to a measure defined for a wider class of sets free of such a restriction.

2. Extension of a measure. Suppose G is a Hausdorff, complete topological additive group with the unit element 0 and ν a G-valued *measure* [1] on a *pseudo-σ-ring* [1] $\mathcal{S}$ of subsets of a fixed set M.

Let Σ be the class of all the pairs $(\mathcal{T}, \lambda)$ such that

1) $\mathcal{T}$ is a pseudo-σ-ring of subsets of M containing $\mathcal{S}$ and each set in $\mathcal{T}$ can be written as a countable sum of sets in $\mathcal{S}$.

2) λ is a G-valued measure defined on $\mathcal{T}$ which is an extension of ν.

Now our purpose is to prove the following theorem.

Theorem 1. *There exists a pair* $(\mathcal{T}_0, \lambda_0)$ *in* Σ *such that* $\mathcal{T}_0 \supset \mathcal{T}$ *and* λ_0 *is an extension of* λ *for any pair* $(\mathcal{T}, \lambda)$ *in* Σ.[1]

Moreover we have the next theorem when we denote by $\mathcal{L}$ the class of all sets L of the form $L=\bigcup_{i=1}^{\infty} X_i$, $X_i \in \mathcal{S}$, $i=1, 2, \cdots$, having the following property: if $X_i \in \mathcal{S}$, $i=1, 2, \cdots$, $X_i \uparrow L$ as $i \to \infty$ and if $Y_j \in \mathcal{S}$, $j=1, 2, \cdots$, $Y_j \uparrow L$ as $j \to \infty$, then, for any neighbourhood U of 0, there is a positive integer n such that $\nu(X_i)-\nu(Y_j) \in U$ for any $i, j \geqq n$.

Theorem 2. *The pseudo-σ-ring* $\mathcal{T}_0$ *in Theorem* 1 *coincides with the class* $\mathcal{L}$.

We are now going to give a proof of Theorem 1 and Theorem 2.

Denote by $\mathcal{M}$ and $\mathcal{K}$ the class of all the subsets of M and the

1) Obviously such a pair $(\mathcal{T}_0, \lambda_0)$ is unique if it exists.

class $\left\{\bigcup_{i=1}^{\infty} X_i \mid X_i \in \mathcal{S},\ i=1, 2, \cdots\right\}$, respectively. Then $\mathcal{M}$ becomes a ring in the algebraic sense when we define, for X, Y in $\mathcal{M}$, $X+Y$, and XY by $(X-Y)\cup(Y-X)$ and $X\cap Y$, respectively.

The fact that $\mathcal{S}$ is a pseudo-σ-ring implies the following two lemmas.

Lemma 1. *$\mathcal{K}$ is a subring of $\mathcal{M}$ satisfying the condition: if $X_i \in \mathcal{K}$, $i=1, 2, \cdots$, then $\bigcup_{i=1}^{\infty} X_i \in \mathcal{K}$.*

Lemma 2. *$\mathcal{S}$ is an ideal of $\mathcal{K}$.*

Further we have

Lemma 3. *$\mathcal{L}$ is an ideal of $\mathcal{K}$.*

Proof. For the proof of this lemma, as is easily seen, it is sufficient to show that

1) if a set $K \in \mathcal{K}$ is contained in some set $L \in \mathcal{L}$, then K belongs to $\mathcal{L}$,

2) if L, L' are sets in $\mathcal{L}$ such that $L \cap L' = \phi$, then $L \cup L' \in \mathcal{L}$.

Let us prove 1). Suppose $X_i \uparrow K$ and $Y_i \uparrow K$ as $i \to \infty$, where X_i, $Y_i \in \mathcal{S}$, and let U be a neighbourhood of 0. Then our assertion is that there exists a positive integer n such that $\nu(X_i)-\nu(Y_j) \in U$ for any $i, j \geqq n$. Suppose this is false. Then inductively we can choose, for each $k=1, 2, \cdots$, i_k and j_k such that $\max(i_k, j_k) < \min(i_{k+1}, j_{k+1})$ and $\nu(X_{i_k})-\nu(Y_{j_k}) \notin U$. Writing $Z=L-K$, we have $Z \in \mathcal{K}$ so that we have $Z_i \in \mathcal{S}$, $i=1, 2, \cdots$, such that $Z_i \uparrow Z$ as $i \to \infty$. Further we have $X'_k \uparrow L$ and $Y'_k \uparrow L$ as $k \to \infty$ when we denote the sets $X_{i_k} \cup Z_k$ and $Y_{j_k} \cup Z_k$, which belong to $\mathcal{S}$, by X'_k and Y'_k, respectively. Since $L \in \mathcal{L}$, this implies that there is a positive integer n such that $\nu(X'_i)-\nu(Y'_j) \in U$ for any $i, j \geqq n$. On the other hand we have $\nu(X'_n)-\nu(Y'_n)=\nu(X_{i_n} \cup Z_n)-\nu(Y_{j_n} \cup Z_n)=\nu(X_{i_n})-\nu(Y_{j_n}) \notin U$. This is a contradiction.

The proof of 2) is as follows. Let X_i, Y_i be sets in $\mathcal{S}$ such that $X_i \uparrow L \cup L'$ and $Y_i \uparrow L \cup L'$ as $i \to \infty$, and let U be a neighbourhood of 0. Writing $Z_i = X_i \cap L$, $W_i = Y_i \cap L$, $Z'_i = X_i \cap L'$, and $W'_i = Y_i \cap L'$, for $i=1, 2, \cdots$, we have Z_i, W_i, Z'_i, and $W'_i \in \mathcal{S}$, which follows from the fact that $\mathcal{S}$ is an ideal of $\mathcal{K}$, and we have $Z_i \uparrow L$, $W_i \uparrow L$, $Z'_i \uparrow L'$, and $W'_i \uparrow L'$ as $i \to \infty$. Let V be a neighbourhood of 0 such that $2V(=V+V) \subset U$. Then, L and L' being sets in $\mathcal{L}$, there are integers l, m such that $\nu(Z_i)-\nu(W_j) \in V$ for any $i, j \geqq l$ and $\nu(Z'_s)-\nu(W'_t) \in V$ for any $s, t \geqq m$. Putting $n=\max(l, m)$, we obtain an integer n such that $\nu(X_i)-\nu(Y_j) \in U$ for any $i, j \geqq n$. This completes the proof of 2) and hence that of Lemma 3.

Corollary. *$\mathcal{L}$ is a pseudo-σ-ring containing $\mathcal{S}$.*

Lemma 4. *The measure ν is uniquely extended to a G-valued* *measure* ν' *defined on the pseudo-σ-ring* $\mathcal{L}$.

Proof. For any set L in $\mathcal{L}$, there is a sequence X_i, $i=1, 2, \cdots$, of sets in $\mathcal{S}$ such that $X_i \uparrow L$ as $i\to\infty$ so that we may define an element $\nu'(L)$ of G as the limit of the sequence $\nu(X_i)$, $i=1, 2, \cdots$. That the sequence has one and only one limit element and that the limit is independent of the choice of X_i's such that $X_i \uparrow L$ as $i\to\infty$ are obvious. Thus we can define a map ν' of $\mathcal{L}$ into G, which is evidently an extension of ν.

To prove that the map ν' is a measure, we have to show that 1) $\nu'(X\cup Y)=\nu'(X)+\nu'(Y)$ for $X, Y\in\mathcal{L}$ such that $X\cap Y=\phi$ and 2) $\nu'(X_i)\to 0$ as $i\to\infty$ for $X_i\in\mathcal{L}$, $i=1, 2, \cdots$, such that $X_i \downarrow \phi$ as $i\to\infty$.

For the proof of 1), let us consider sets X_i and Y_i, $i=1, 2, \cdots$, in $\mathcal{S}$ such that $X_i \uparrow X$ and $Y_i \uparrow Y$ as $i\to\infty$. The definition of the map ν' implies that $\nu(X_i)\to\nu'(X)$ and $\nu(Y_i)\to\nu'(Y)$ as $i\to\infty$, while the formula $X_i\cup Y_i \uparrow X\cup Y$ as $i\to\infty$ implies that $\nu(X_i\cup Y_i)\to\nu'(X\cup Y)$ as $i\to\infty$. Since $\nu(X_i\cup Y_i)=\nu(X_i)+\nu(Y_i)$, $i = 1, 2, \cdots$, it follows that $\nu'(X\cup Y)=\nu'(X)+\nu'(Y)$.

The proof of 2) is as follows. Let U be a neighbourhood of 0. Then our statement is that there exists a positive integer n such that $\nu'(X_i)\in U$ for any $i\geqq n$. Put $Y_i=X_1-X_i$ for $i=1, 2, \cdots$. Since $Y_i\in\mathcal{L}$ we have a sequence Y_{ij}, $j=1, 2, \cdots$, of sets in $\mathcal{S}$ such that $Y_{ij}\uparrow Y_i$ as $j\to\infty$, for each i. It may be assumed for each j that $Y_{ij}\subset Y_{i'j}$ if $i<i'$ because $Y_i\uparrow X_1$ as $i\to\infty$. For each i, we have $\nu(Y_{ij})\to\nu'(Y_i)$ as $j\to\infty$ and therefore we can choose an integer j_i such that $\nu(Y_{ij_i})-\nu'(Y_i)\in V$, where V is a neighbourhood of 0 such that $2V\subset U$. We may assume that $j_i<j_{i'}$ if $i<i'$. When we put $Z_i=Y_{ij_i}$, for $i=1, 2, \cdots$, the formula $Z_i\uparrow X_1$ as $i\to\infty$ yields $\nu(Z_i)\to\nu'(X_1)$ as $i\to\infty$. Hence there exists a positive integer n such that $\nu'(X_1)-\nu(Z_i)\in V$ for any $i\geqq n$. Let us show that $\nu'(X_i)\in U$ for any $i\geqq n$, which completes the proof of 2) and consequently shows that ν' is a measure. For any integer $i\geqq n$, we may write $\nu'(X_1)-\nu(Z_i)\in V$ and $\nu(Z_i)-\nu'(Y_i)=\nu(Y_{ij_i})-\nu'(Y_i)\in V$. The formula $\nu'(X_1)=\nu'(X_i\cup Y_i)=\nu'(X_i)+\nu'(Y_i)$, which follows from the already proved proposition 1), implies that $\nu'(X_i)=\nu'(X_1)-\nu'(Y_i)=\{\nu'(X_1)-\nu(Z_i)\}+\{\nu(Z_i)-\nu'(Y_i)\}\in 2V\subset U$.

Thus it is proved that there exists at least one G-valued measure ν' defined on $\mathcal{L}$ which is an extension of ν. The uniqueness of such a measure ν' is obvious and this completes the proof of Lemma 4.

Proof of Theorem 1 and Theorem 2. Put $\mathcal{T}_0=\mathcal{L}$ and $\lambda_0=\nu'$, where ν' is the measure defined in Lemma 4. Then the pair $(\mathcal{T}_0, \lambda_0)$

belongs to the class Σ and it is easily seen that $\mathcal{T}_0 \supset \mathcal{T}$ and λ_0 is an extension of λ for any pair $(\mathcal{T}, \lambda)$ in Σ (cf. [1] *p.* 331, footnote 2)). Hence Theorem 1 is proved and the uniqueness of the pair $(\mathcal{T}_0, \lambda_0)$ in Theorem 1 implies that Theorem 2 is true.

The following proposition follows immediately from Lemma 2.

Proposition 1. *$\mathcal{S}$ is an ideal of $\mathcal{T}_0 = \mathcal{L}$.*

We shall close this paper with

Theorem 3. *If the measure ν is of bounded variation,*[2] *so is the measure λ_0.*

Proof. Let X be a set in $\mathcal{T}_0$ and let U be a neighbourhood of 0. Now there is a neighbourhood V of 0 such that $3V \subset U$.

We assert that there exists a set Y in $\mathcal{S}$ contained in X such that $\nu(Z) \in V$ for any set Z in $\mathcal{S}$ contained in $X - Y$. Otherwise, we have a disjoint sequence Y_i, $i=1, 2, \cdots$, of sets in $\mathcal{S}$ such that $Y_i \subset X$, $\nu(Y_i) \notin V$ for $i=1, 2, \cdots$. Then, putting $Y'_k = \bigcup_{i=1}^{k} Y_i$ and $Y' = \bigcup_{k=1}^{\infty} Y'_k$, we have $Y' \in \mathcal{T}_0$, $Y'_k \in \mathcal{S}$, $k=1, 2, \cdots$, and $Y'_k \uparrow Y'$ as $k \to \infty$. Hence we get $\nu(Y'_k) \to \lambda_0(Y')$ as $k \to \infty$ contrary to the formula $\nu(Y'_k) - \nu(Y'_{k-1}) = \nu(Y'_k - Y'_{k-1}) = \nu(Y_k) \notin V$ for any $k > 1$. This shows that our assertion is true.

Consider such a set Y. By the assumption that ν is of bounded variation, there is a positive integer n satisfying the condition: if Z_i, $i=1, 2, \cdots, n$, are disjoint sets in $\mathcal{S}$ contained in Y, then $\nu(Z_{i_0}) \in V$ for some integer $i_0 \leqq n$.

Now suppose X_i, $i = 1, 2, \cdots, n$, are disjoint sets in $\mathcal{T}_0$ contained in X. For the proof of this theorem, it is sufficient to prove that there is an integer $i_0 \leqq n$ such that $\lambda_0(X_{i_0}) \in U$. Since X_i is a set in $\mathcal{T}_0$, there is a set Y_i in $\mathcal{S}$ contained in X_i such that $\lambda_0(X_i) - \nu(Y_i) \in V$, for each $i=1, 2, \cdots, n$. Writing $Z_i = Y_i \cap Y$, we have a disjoint sequence Z_i, $i=1, 2, \cdots, n$, of sets in $\mathcal{S}$ contained in Y. By the definition of the integer n, there is an integer $i_0 \leqq n$ such that $\nu(Z_{i_0}) \in V$. Becasuse $Y_{i_0} - Z_{i_0}$ is a set in $\mathcal{S}$ contained in $X - Y$, we have $\nu(Y_{i_0} - Z_{i_0}) \in V$ and therefore we have $\lambda_0(X_{i_0}) = \{\lambda_0(X_{i_0}) - \nu(Y_{i_0})\} + \nu(Y_{i_0}) \in V + \nu(Y_{i_0} - Z_{i_0}) + \nu(Z_{i_0}) \subset 3V \subset U$. This proves the theorem.

Reference

[1] M. Takahashi: On topological-additive-group-valued measures. Proc. Japan Acad., **42**, 330-334 (1966).

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2) The definition is given in [1].