192. On Axiom Systems of Propositional Calculi. XXII

By Shôtarô TANAKA

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Prof. K. Iséki has given an algebraic formulation of the Tarski-Bernays axioms system of restricted propositional calculus. The original axioms are the following:

(1) *CpCqp*,

(2) *CCpqCCqrCpr*,

(3) *CCCpqpp*,

Further he has proved that this system implies some axiom systems of the restricted propositional calculus by using his algebraic technique (see, [2]). Among them there is the Lukasiewicz axioms system, i.e.,

(3) *CpCqp*,

(4) *CCCCpqrsCCqsCps*.

On the other hand, there is a single axiom of this propositional calculus:

(5) *CCCpCqpCCCCCrstuCCsuCruvv*.

In his paper [1], Prof. K. Iséki has proved that (1), (2), and (3) imply (5).

In this paper, we shall prove that the single axiom (5) implies the Lukasiewicz axioms (3), (4), and further (3), (4) imply the Tarski-Bernays axioms. We only use the rules of substitution and detachment.

1 *CCCpCqpCCCCCrstuCCsuCruvv*.

1 *p/CCCCCpqsrCCqrCCprCCCCpCqpCCCCCrstuCCsu CruvvCCCCpqsrCCqrCpr*, *q/CCCCpqsrCCqrCpr*, *r/p*, *s/q*, *t/s*, *u/r*, *v/CCCCpCqpCCCCCrstuCCsuCruvv CCCCpqsrCCqrCpr* **C*1 *p/CCCpqsrCCqrCpr*, *q/CCCpCqpCCCCCrstuCCsuCruvv*, *r/p*, *s/q*, *t/s*, *u/r*, *v/CCCCCpqsrCCqrCprCCCCpCqpCCCCCrstuCCsu CruvvCCCCpqsrCCqrCpr*—*C*1—2,

2 *CCCCpqsrCCqrCpr*.

2 *p/Cpq*, *q/s*, *s/r*, *r/CCqrCpr* **C*2—3,

3 *CCsCCqrCprCCpqCCqrCpr*.

3 *s/CCpqsr* **C*2—4,

4 *CCpqCCqrCpr*.

2 *q/Cqp*, *s/CCCCrstuCCsuCruv*, *r/v* **C*1—5,

5 *CCCqpvCpv*.

5 q/Cqp, p/v, v/Cpv $*C5$—6,

6 $CvCpv$.

2 s/r, r/Cqr $*C5$ q/p, p/q, v/r—7,

7 $CCqCqrCpCqr$.

7 r/q, $p/CvCpv$ $*C6$ v/q, p/q—$C6$—8,

8 Cqq.

2 $r/CCrst$ $*C8$ $q/CCrst$—9,

9 $CCsCCrstCrCCrst$.

9 r/p, s/q, t/q $*C6$ v/q, p/Cpq—10,

10 $CpCCpqq$.

4 $q/CCpqq$ $*C10$—11,

11 $CCCCpqqrCpr$.

4 $p/CpCqr$, $q/CCCqrrCpr$, $r/CqCpr$ $*C4$ q/Cqr—$C11$ p/q, q/r, r/Cpr—12,

12 $CCpCqrCqCpr$.

12 $p/CqCpr$, q/Cpp, r/Cqr $*C7$ p/Cpp—13,

13 $CCqCqrCqr$.

4 $p/CCqpv$, q/Cpr $*C5$—14,

14 $CCCpvrCCCqpvr$.

4 $p/CCpvr$, $q/CCCqpvr$, r/s $*C14$—15,

15 $CCCCCqpvrsCCCpvrs$.

15 $s/CCprCqr$ $*C2$ p/q, q/p, s/v—16,

16 $CCCpvrCCprCqr$.

4 $p/CCpvr$, $q/CCprCqr$, r/s $*C16$—17,

17 $CCCCprCqrsCCCpvrs$.

17 q/Cpr, $s/CCprr$ $*C13$ q/Cpr—18,

18 $CCCpvrCCprr$.

18 $r/CCCpvqq$ $*C10$ p/Cpv—19,

19 $CCpCCCpvqqCCCpvqq$.

19 q/p $*C6$ v/p, $p/CCpvp$—20,

20 $CCCpvpp$.

Theses 4, 6, and 20 are axioms of the Tarski-Bernays system. Therefore the proof is complete.

References

[1] K. Iséki: On axiom systems of propositional calculi. XXI. Proc. Japan Acad., **42**, 441-442 (1966).

[2] Y. Imai and K. Iséki: On axiom systems of propositional calculi. XIV. Proc. Japan Acad., **42**, 19-22 (1966).