190. *Axiom Systems of B-Algebra*

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In his note [1] Kiyoshi Iséki gave an algebraic formulation of propositional calculi and he defined B-algebra.

Other characterisations of B-algebra are given by K. Iséki, Y. Arai, and K. Tanaka (see [2]-[5]).

Let $\langle X, 0, *, \sim\rangle$ be an algebra where 0 is an element of a set X, $*$ is a binary operation and $\sim$ is an unary operation on X. We write $x\leq y$ for $x*y=0$, and $x=y$ for $x\leq y$ and $y\leq x$.

The axiom system of B-algebra is given by (see [2])

$H1$. $x*y\leq x$,

$H2$. $(x*y)*z\leq(x*z)*y$,

$H3$. $(x*y)*(x*z)\leq z*y$,

$H4$. $x*(\sim y)\leq y$,

$H5$. $x*(x*(\sim y))\leq x*y$,

$H6$. $0\leq x$.

In this note we shall show that a B-algebra is characterized by the following axiom system.

$B1$. $(x*y)*z\leq x$,

$B2$. $x*y\leq\sim y$,

$B3$. $(x*(y*z))*(x*y)\leq x*(\sim z)$,

$B4$. $0\leq x$.

Lemma 1. $H\Rightarrow B$.

In $H2$, put $z=\sim y$, then by $H4$, we have

(1) $$x*y\leq\sim y,$$

which is axiom $B2$.

In $H3$, put $x*z=z*y=0$, then

(2) $$x\leq y,\ y\leq z \text{ imply } x\leq z.$$

In $H1$, put $x=x*y$, $y=z$, then by $H1$ we have

(3) $$(x*y)*z\leq x*y.$$

By (3), $H1$ and (2) we have

(4) $$(x*y)*z\leq x$$

which is axiom $B1$.

Put $y=y*z$, $z=x*y$ in $H2$, then, we have

(5) $$(x*(y*z))*(x*y)\leq(x*(x*y))*(y*z).$$

Let us put $x=x*z$, $y=y*z$, $z=x*y$ in $H2$, then

$$((x*z)*(y*z))*(x*y)\leq((x*z)*(x*y))*(y*z).$$

The right side is equal to 0 by $H3$, hence we have

(6) $$(x*z)*(y*z)\leq x*y.$$

Let $x*y=0$, then by (6) we have

(7) $$x\leq y \text{ imply } x*z\leq y*z.$$

In [2], K. Iséki proves that H implies the followings:

(8) $$x=\sim(\sim x),$$

(9) $$x*y\leq(\sim y)*(\sim x).$$

By $H5$ and (8) we have

(10) $$x*(x*y)\leq x*(\sim y).$$

By (10) and (7) we have

(11) $$(x*(x*y))*(y*z)\leq(x*(\sim y))*(y*z).$$

By (9) and (8) we have

(12) $$x*(\sim y)\leq y*(\sim x).$$

By (12) and (7) we have

(13) $$(x*(\sim y))*(y*z)\leq(y*(\sim x))*(y*z).$$

Let us put $x=y$, $y=\sim x$ in $H3$, then

(14) $$(y*(\sim x))*(y*z)\leq z*(\sim x).$$

By (14), (12), and (2) we have

(15) $$(y*(\sim x))*(y*z)\leq x*(\sim z).$$

By (5), (11), (13), (15), and (2) we have

(16) $$(x*(y*z))*(x*y)\leq x*(\sim z)$$

which is axiom $B3$.

Therefore we complete the proof of Lemma 1.

Lemma 2. $B\Rightarrow H$.

From $B3$ and $B2$ we have

(17) $$(x*z)*(y*z)\leq(x*z)*y.$$

In (17), put $x=x*y$, $y=x$, then by $B1$ we have

(18) $$(x*y)*z\leq x*z.$$

By (18) and $B2$ we have

(19) $$(x*y)*z\leq\sim y.$$

In $B3$ put $x=(x*y)*z$, $y=x*z$, $z=y$ then

$$(((x*y)*z)*((x*z)*y))*(((x*y)*z)*(x*z))\leq((x*y)*z)*(\sim y).$$

The right side is equal to 0 by (19), and further the second term of the left side is 0 by (18), hence we have the following.

(20) $$(x*y)*z\leq(x*z)*y,$$

which is axiom $H2$.

In (20) put $x=x*y$, $y=x*z$, then by (18) we have

(21) $$(x*y)*(x*z)\leq z.$$

We substitute $(x*y)*(x*z)$ for x, z for y, y for z in $B3$, then

$$(((x*y)*(x*z))*(z*y))*(((x*y)*(x*z)*z)\leq((x*y)*(x*z))*(\sim y).$$

By the formula (19), the right side is equal to 0, and by the formula above, the second term of the left side is 0, hence we have $H3$

(22) $$(x*y)*(x*z)\leq z*y.$$

If we substitute $\sim y$ for y, y for z in (20), then by $B2$ we have $H4$

(23) $$x*(\sim y)\leq y.$$

In (20) put $x=x*z$, $y=(x*z)*y$, $z=y*z$, then

$$((x*z)*((x*z)*y))*(y*z)\leq((x*z)*(y*z))*((x*z)*y).$$

The right side is equal to 0 by (17), hence

(24) $$(x*z)*((x*z)*y)\leq y*z.$$

Next we substitute $z*(\sim y)$ for y, y for z, then by (23) we have

(25) $$x*y\leq(x*y)*(z*(\sim y)).$$

By $B1$ we have the following relation

(26) $$(x*y)*(z*(\sim y))\leq x.$$

By (22) we have

(27) $$x\leq y,\ y\leq z \text{ imply } x\leq z.$$

By (25), (26), and (27) we have the following

(28) $$x*y\leq x$$

which is axiom $H1$.

In (20) put $y=x$, then

(29) $$x*x\leq z.$$

By (29) and the definition of equality, we have

(30) $$x*x=0.$$

In (20) if we put $x=x*(z*y)$, $y=x*(\sim y)$, $z=x*z$ and use $B3$, then we have

(31) $$(x*(z*y))*(x*(\sim y))\leq x*z.$$

By (31), (30) we have

(32) $$x*(x*y)\leq x*(\sim y).$$

In (32) put $x=x*(\sim(\sim x))$, $y=x$, then by (23), (28), we have

(33) $$x\leq\sim(\sim x).$$

If we substitute $x*(\sim y)$ for z in (22), then by (23) we have the following

(34) $$x*y\leq x*(x*(\sim y)).$$

Let us put $x=\sim(\sim x)$, $y=\sim(\sim x)$, $z=\sim x$ in $B3$ then we have

$$((\sim(\sim x))*((\sim(\sim x))*(\sim x)))*((\sim(\sim x))*(\sim(\sim x)))\leq (\sim(\sim x))*(\sim(\sim x)).$$

Here $(\sim(\sim x))*(\sim(\sim x))=0$ by (30), then we have

(35) $$\sim(\sim x)\leq(\sim(\sim x))*(\sim x).$$

In (34) put $x=\sim(\sim x)$, $y=x$, then by the formula above we have

(36) $$\sim(\sim x)\leq x.$$

By (33), (36) and the definition of equality, we have

(37) $$x=\sim(\sim x).$$

In (32) if we put $y=\sim y$ and use (37) we have

(38) $$x*(x*(\sim y))\leq x*y.$$

which completes the proof of $B\Rightarrow H$.

Hence we have the following

Theorem. *Any B-algebra is characterized by the axiom system* $B1-B4$.

References

[1] K. Iséki: Algebraic formulation of propositional calculi. Proc. Japan Acad., **41**, 803-807 (1965).

[2] ——: Axiom systems of B-algebra. Proc. Japan Acad., **41**, 808-811 (1965).

[3] Y. Arai and K. Iséki: Axiom systems of B-algebra II. Proc. Japan Acad., **41**, 908-910 (1965).

[4] K. Iséki: A characterization of Boolean algebra. Proc. Japan Acad., **41**, 893-897 (1965).

[5] K. Tanaka: On axiom systems of propositional calculi. XI. Proc. Japan Acad., **41**, 898-900 (1965).