229. *Algebraic Formulations of Propositional Calculi with Variable Forming Functors*

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(Comm. by Kinjirô KUNUGI, M.J.A., Nov. 12, 1966)

In my paper [1], we consider an algebraic formulation of propositional calculi. In this note, we shall give an idea of an algebraic formulation of propositional calculus with variable proposition forming functors in S. Lésniewski's protothetic. Such an algebraic formulation will become very important ideas to give a simple axiom system of mathematical theory. We shall explain my ideas using some results obtained by J. Lukasiewicz [2], [3], C. A. Meredith [4], and A. N. Prior [5].

Let $p, q, r, \cdots$ be any significant expressions and suppose $p, q, r, \cdots$ do not contain δ. Then the characteristic property of the definition is written in the form of

(1) $$C\delta p\delta q$$

by following J. Lukasiewicz [2], [3]. Then by our notation mentioned in (1), Thesis (1) is expressed by

(2) $$f(q)\leqslant f(p),$$

where f is *any* variable functor (For detail on $\leqslant$, see (1)).

J. Lukasiewicz proved an important result: $C\delta q\delta p$. In our notation, this is proved as follows.

Let

$$f(x)=(g(p)*g(q))*(g(x)*g(p)),$$

then by (2), we have

(3) $$(g(p)*g(q))*(g(q)*g(p))\leqslant(g(p)*g(q))*(g(p)*g(p)).$$

On the other hand, let

$$f(x)=g(p)*g(x),$$

then by (2), we have

(4) $$g(p)*g(q)\leqslant g(p)*g(p).$$

By (3) and (4), we have

(5) $$g(p)*g(q)\leqslant g(q)*g(p).$$

(2) means $g(q)\leqslant g(p)$, hence

(6) $$g(p)\leqslant g(q),$$

and we have $f(p)\leqslant f(q)$, since g is arbitrary.

As an example, we take up a *formula* $C\delta CqpC\delta q\delta p$, i.e.

(7) $$f(p)*f(q)\leqslant f(p*q).$$

Let $f(x)=x*x$, then we have

(8) $$(p*p)*(q*q)\leqslant(p*q)*(p*q).$$

On the other hand, put $f(x)=x$, then $p*q\leqslant p*q$. This and (8) imply

(9) $$p*p\leqslant q*q.$$

Since q is arbitrary, we have

(10) $$p\leqslant p.$$

As in [1], we consider zero element 0, then we have $0\leqslant p$, i.e. $0*p=0$ for every p.

Let $f(x)=x*r$ in (7), then

(11) $$(p*r)*(q*r)\leqslant(p*q)*r.$$

As another example, we consider a formula $C\delta CpqCp\delta q$, i.e.

(12) $$f(q)*p\leqslant f(q*p).$$

Let $f(x)=x$ in (12), then we have

(13) $$q*p\leqslant q*p.$$

Put $f(x)=x*x$ in (12), then $(q*q)*p\leqslant(q*p)*(q*p)=0$, hence $q*q\leqslant p$. Since p is arbitrary, we

(14) $$p\leqslant p,\ \text{i.e.}\ p*p=0.$$

Let $f(x)=x*r$ in (12), then

(15) $$(q*r)*p\leqslant(q*p)*r.$$

Let $q=p$, $r=q$ in (15), then $(p*q)*p\leqslant(p*p)*q=0*q=0$, hence

(16) $$p*q\leqslant p.$$

Remark. There are some delicate problems among syllogism, equal $=$ and order symbol $<$. For these problems, see [1].

References

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