224. Integration on Locally Compact Spaces Generated by Positive Linear Functionals Defined on the Space of Continuous Functions with Compact Support and the Riesz Representation Theorem.*) I

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A non-empty family V of sets of an abstract space X is called a *prering* if the following condition is satisfied: if $A, B \in V$ then $A \cap B \in V$ and there exists disjoint sets $C_1, \cdots, C_k \in V$ such that $A \backslash B = C_1 \cup \cdots \cup C_k$.

A function μ from a prering V into a Banach space Z is called a vector volume if it satisfies the following condition: for every countable family of disjoint sets $A_t \in V (t \in T)$ such that

(a) $$A = \bigcup_T A_t \in V$$

we have $\mu(A) = \sum_T \mu(A_t)$, where the last sum is convergent absolutely and the variation of the function μ, that is, the function

$$|\mu|(A) = sup\left\{\sum_T |\mu(A_t)|\right\}$$

is finite for every set $A \in V$, where the supremum is taken over all possible decompositions of the set A into the form (a). A volume is called positive if it takes on only non-negative values. If μ is a volume then its variation $|\mu|$ is a positive volume.

If v is a volume on a prering V of subsets of a space X then the triple (X, V, v) is called a volume space.

Let R be the space of reals and Y, Z, W be Banach spaces. Denote by X the space of all bilinear continuous operators u from the space $Y \times Z$ into the space W. Norms of elements in the spaces Y, Y', Z, W, U will be denoted by $|\ \ |$.

In the paper [1] has been presented an approach to the theory of the space $L(v, Y)$ of Lebesgue-Bochner summable functions generated by a positive volume v. The construction was not based on measure or on measurable functions. It allowed us to prove the basic structure theorems of the space of summable functions and at the same time to develop the theory of an integral of the form $\int u(f, d\mu)$, where u denotes a bilinear continuous operator from $U = L(Y, Z; W), f \in L(v, Y)$, and μ is a finitely additive func-

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tion from the prering V to the Banach space Z, dominated by the volume cv for some constant c, that is, such that the estimation holds

$$|\mu(A)|\leq cv(A) \qquad \text{for all } A \in V.$$

The construction of the theory of Lebesgue-Bochner measurable functions and the theory of measure corresponding to the approach of [1] has been developed in [3]. This approach permitted us to simplify in [5] the construction and the theory of integration on locally compact spaces.

In the paper [2] has been presented an approach to the theory of integration generated by a positive linear functional defined on any linear lattice of real-valued functions. The approach was based only on the results of [1].

Using the results of [2] we shall show in §1 of this paper how one may develop the theory of integration on locally compact spaces generated by positive linear functionals defined on the space C_0 of all continuous functions f with compact support from a locally compact space X into the space of reals R.

We will say that the set $A\subset X$ is bounded if its closure is compact. The family V_1 of all sets of the form $A=G_1\backslash G_2$, where G_i are open, bounded sets, forms a prering which will be called the *Borel prering*.

The family V of all sets of the form $A=G_1\backslash G_2$, where G_i are open bounded F_σ sets, forms a prering which will be called the *Baire prering*.

The smallest sigma-ring containing the Baire prering or the Borel prering will be called respectively the *Baire or the Borel ring*. It is easy to see that the Borel ring is the smallest sigma-ring containing all bounded open sets, and the Baire ring is the smallest sigma-ring containing all open bounded F_σ sigma sets.

A real-valued function v on a family of sets V of a topological space X is called regular if the following conditions are satisfied

$$v(A)=sup\,\{v(E): \bar{E}\subset A,\ E \in V\}$$

and

$$v(A)=inf\,\{v(E): A\subset int\,E,\ E \in V\}$$

for all sets $A \in V$.

A positive volume or a positive measure defined on the Borel prering or the Borel ring, respectively, is called *Borel volume or Borel measure*, respectively, if it is regular.

A positive volume or a positive measure defined on the Baire prering or the Baire ring, respectively, is called the *Baire volume or the Baire measure*. It is easy to prove that every Baire volume and therefore every Baire measure is regular. (See for example [5]).

The main result of § 1 is that every positive linear functional on the space C_0 can be represented by means of integral functionals with respect to the Baire or the Borel volume.

This permits us to obtain in § 2 representations of positive linear functionals on the space C_0 by means of integrals generated by Borel or Baire measures.

§ 1. Representations by means of integrals generated by volumes. Let μ be a volume from the prering V of subsets of X into the space R of reals. In this paper the integral $\int f\,d\mu$ we shall understand as the integral $\int u(f, d\mu)$ corresponding to the bilinear operator $u(y, z)=yz$ for $y \in Y, z \in R$ and considered on the space $L(|\mu|, Y)$.

Let X be a locally compact Hausdorff space and let C_0 be the space of real-valued continuous functions with compact support on it. It is easy to see that the space C_0 is linear. Since for every function $f \in C_0$, the function $|f|(x)=|f(x)|$ $(x \in X)$ also belongs to the space C_0 therefore the space C_0 is a linear lattice. For equivalent conditions for a linear space to be a linear lattice cf. Prop. 1, [2].

A real-valued functional J defined on a linear lattice L is called *positive* if $Jf \geq 0$ whenever $f \in L$ and $f(x) \geq 0$ for all $x \in X$.

A positive linear functional J on a linear lattice L is called a *Daniell functional* if $J(f_n) \to 0$, whenever $f_n \in L$ and the sequence $f_n(x)$ converges decreasingly to 0 for every $x \in X$. For equivalent conditions to this one see Prop. 2 [2].

Proposition 1. *Every linear positive functional on the space C_0 is a Daniell functional.*

The proof is evident. It makes use of the theorem that every sequence of continuous functions decreasingly convergent to zero on a compact set, converges uniformly to zero. See for example Bourbaki [10].

Let J be a linear positive functional on the space C_0. Let (X, V, v) be the volume space generated by J defined in [2].

Let $L_v = L(v, R)$ and $\int_v$ be the coresponding integral functional defined by $\int_v f = \int f dv$ for all $f \in L_v$. We shall write $J \subset \int_v$ if the functional $\int_v$ is an extension of the functional J. The same notation will be used for volumes and measures.

Notice that the space C_0 satisfies the *Stone condition*: $f \cap 1 \in C_0$ for all $f \in C_0$. Therefore, according to Th. 4 [2] we have $J \subset \int_v$.

A more precise description gives the following

Theorem 1. *The class of volumes generated by Daniell functionals defined on C_0 coincides with the class of Baire volumes. If J is a Daniell functional on C_0 and v the corresponding Baire volume then* $J\subset\int_v$.

We also have

Theorem 2. *For every Daniell functional J on C_0 there exists a unique Borel volume v_1 such that* $J\subset\int_{v_1}$.

§ 2. Representation by means of measures. If μ is a measure on a sigma-ring M. Put $P=\{A\in M: \mu(A)<\infty\}$. Notice that this family of sets forms a prering and the function w being the restriction of the measure to P is a volume.

Let $\bar{\mu}$ be the completion of the measure μ, that is, the smallest measure η such that $\mu\subset\eta$.

It is easy to prove that the space $L(w, Y)$ of Lebesgue-Bochner summable functions as developed in [1] coincides with the space $L(\bar{\mu}, Y)$ generated by the measure $\bar{\mu}$ by means of any other classical construction and we have $\|\ \|_w=\|\ \|_{\bar{\mu}}$, $\int f d\mu=\int f dv$ for all $f\in L(v, Y)$. For detail see [7], [8].

Let J be a Daniell functional and v be the volume generated by it. Denote by μ_v the corresponding measure as constructed in [5]. Let μ be the restriction of the measure to the Baire ring M.

Similarly, if v_1 is the Borel volume generated by the functional denote by μ_1 the measure obtained on the Borel ring. It follows from the results of [6] that the measure is regular and therefore is a Borel measure.

Using the results of [5] and in particular the result of the last paragraph of that paper we get

Theorem 3. *Let J be a Daniell functional on C_0. Let v, v_1 be the corresponding Baire and Borel volumes and μ, μ_1 their extensions to a Baire and a Borel measure. Then*

$$J=\int_v=\int_\mu\subset\int_{v_1}=\int_{\mu_1}.$$

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