222. On Branching Semi-Groups. II

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(Comm. by Kinjirô KUNUGI, M.J.A., Nov. 12, 1966)

We have discussed in [4] and [5] on the methods of the construction of a branching Markov process. The purpose of this paper is to give another analytic method of construction based on S-equation.[1] To do this we shall first construct a solution of S-equation with a initial value by the usual method of successive approximation and then we shall define a branching semi-group with the aid of these solutions. It will turn out that this semi-group coincides with that constructed in [5] by the method of Moyal [7]. This fact follows from a result of [5] that the semi-group constructed in [5] by the method of Moyal is a branching semi-group. (the proof depends essentially on the Theorem 1 of [2].) But we shall give still another proof based on the uniquness of the solution of the forward equation.[2] This may be considered as a generalization of a method of Harris [6] to prove that (π_{ij}, q_j)-minimal Markov chain on $Z^+=\{0, 1, 2, 3, \cdots\}$ where $\pi_{ij}=p_{j-i+1}$ and $q_j=jb$[3] is a branching Markov process, i.e. its transition probability $\{p_{ij}(t)\}$ satisfies

$$\sum_{j=0}^{\infty} p_{ij}(t)s^j=\left[\sum_{j=0}^{\infty} p_{1j}(t)s^j\right]^i, \qquad \text{for every } 0<s\leqq 1.$$

Let S be a compact metrizable space and $\boldsymbol{S}=\bigcup_{n=0}^{\infty} S^n \cup \{\varDelta\}$ be defined as in [2]. Let T_t be a positive strongly continuous semi-group on $\boldsymbol{C}(S)$ such that $T_t 1=1$ and take $k\in \boldsymbol{C}(S)^+$. Let $\mathfrak{G}$ be the infinitesimal generator of T_t in the Hille-Yosida sense and $\mathfrak{D}(\mathfrak{G})$ be the domain of $\mathfrak{G}$. Then it is well known that there exists uniquely a positive strongly continuous semi-group T_t^0 on $\boldsymbol{C}(S)$ such that $T_t^0 1\leqq 1$ and its generator $\mathfrak{G}^0$ is given by

(1) $$\mathfrak{G}^0=\mathfrak{G}-k \quad \text{and} \quad \mathfrak{D}(\mathfrak{G}^0)=\mathfrak{D}(\mathfrak{G}).$$ [4]

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1) Cf. [3]. In the following we use the terminology and the notation of [2], [3], [4], and [5].

2) Cf. [3].

3) $\{p_i\}, i=0, 1, 2, 3, \cdots$ is a given probability sequence and b is a given positive constant.

4) A probabilistic method to obtain T_t^0 from the given T_t and k is the killing defined by the multiplicative functional $\exp\left(-\int_0^t k(x_s)ds\right)$.

Let

$$T_t^0 f(x) = \int_S T_t^0(x, dy) f(y), \qquad x \in S,$$

and set

(2) $$K(x, dsdy) = T_s^0(x, dy) k(y) ds, \qquad x, y \in S,\ s \in [0, \infty).$$

Finally let $\pi(x, d\boldsymbol{y})$, $x \in S$, $\boldsymbol{y} \in \boldsymbol{S} - \{\varDelta\}$ be a non-negative kernell on $S \times (\boldsymbol{S} - \{\varDelta\})$ such that $\pi(x, \boldsymbol{S} - \{\varDelta\}) = 1$ for every $x \in S$ and $F[x; f]$ defined by

(3) $$F[x; f] = \int_{\boldsymbol{S}-\{\varDelta\}} \pi(x, d\boldsymbol{y}) \hat{f}(\boldsymbol{y})$$

belongs to $\boldsymbol{C}(S)$, provided that $f \in C^*(S)^+$.[5]

Lemma 1. *For every* $0 < r < 1$, *we have*

(4) $$\|\hat{f} - \hat{g}\|_{\boldsymbol{S}} \leqq C_r \|f - g\|,$$

and

(5) $$\|\langle f \mid u\rangle - \langle g \mid v\rangle\|_{\boldsymbol{S}} \leqq K_r \|u\| \|f - g\| + L_r \|u - v\|,$$[6]

provided that $f, g, u, v \in \boldsymbol{C}(S)$ *and* $0 \leqq f, g \leqq r$, *where* C_r, K_r, *and* L_r *are positive constants.*

Given $f \in \boldsymbol{C}(S)$ such that $0 \leqq f \leqq 1$, we shall consider the following equation (S-equation)

(6) $$u_t(x) = T_t^0 f(x) + \int_0^t \int_S K(x, dsdy) F[y; u_{t-s}],$$

then we have

Theorem 1. *There exists a unique solution* $u_t(x) \equiv u_t(x; f)$ *of* (6), *provided that* $f \in \boldsymbol{C}^*(S)^+$. *Furthermore, it satisfies*

(i) $$u_t(\cdot) \in C^*(S)^+,$$

(ii) $$\|u_t(\cdot) - f\| \to 0, \qquad (t \to 0),$$

and

(iii) $$u_{t+s}(\cdot; f) = u_t(\cdot; u_s(\cdot; f)).$$

This theorem is proved by the usual method of successive approximation if we note that for every $0 < r < 1$ we have

(7) $$\|F[\cdot; f] - F[\cdot; g]\| \leqq C_r \|f - g\|,$$

provided that $f, g \in \boldsymbol{C}(S)$ and $0 \leqq f, g \leqq r$, where C_r is a positive constant. (7) follows directly from Lemma 1 (4). $u_t(\cdot; f)$ is given as a limit of $u_t^{(n)}$ in $\boldsymbol{C}(S)$, which is defined successively by

$$u_t^{(0)} \equiv 0,$$

$$u_t^{(n)} = T_t^0 f + \int_0^t \int_S K(\cdot, dsdy) F[y; u_{t-s}^{(n-1)}(\cdot)], \qquad n \geqq 1.$$

By virtue of this construction, it is easy to see that for fixed $t > 0$ and $x \in S$ $u_t(x; f)$ is given by

$$u_t(x; f) = \int_{\boldsymbol{S}-\{\varDelta\}} \hat{f}(\boldsymbol{y}) \mu_t^x(d\boldsymbol{y}),$$

where $\mu_t^x(d\boldsymbol{y})$ is a substochastic measure on $\boldsymbol{S} - \{\varDelta\}$. Then we find

5) $\boldsymbol{C}^*(S)^+ = \{f \in \boldsymbol{C}(S), 0 \leqq f < 1\}$.

6) Cf. Definition 2.1 of [3].

that, by Lemma 2.2 of [3], there exists a substochastic kernel $\boldsymbol{T}_t(\boldsymbol{x}, d\boldsymbol{y})$ on $(\boldsymbol{S}-\{\Delta\})\times(\boldsymbol{S}-\{\Delta\})$ such that

$$\widehat{u_t(\cdot\,;f)}(\boldsymbol{x})=\int_{\boldsymbol{S}-\{\Delta\}}\boldsymbol{T}_t(\boldsymbol{x}, d\boldsymbol{y})\hat{f}(\boldsymbol{y}). \tag{8}$$

It is to be noticed that $\{T_t\}$ is uniquely determined by virtue of Lemma 2.1 of [3].

Then we can prove that $\boldsymbol{T}_t$ defines a strongly continuous semi-group on $\boldsymbol{C}_0(\boldsymbol{S})$ by means of (4) in Lemma 1, Lemma 2.1 of [3] and Theorem 1. Moreover the formula (8) proves that $\boldsymbol{T}_t$ is a branching semi-group. Thus we have

Theorem 2. *There exists a unique non-negative strongly continuous branching semi-group $\boldsymbol{T}_t$ on $\boldsymbol{C}_0(\boldsymbol{S})$ such that $u_t(x)=\boldsymbol{T}_t\hat{f}(x)$, $f\in \boldsymbol{C}^*(S)^+$, $x\in S$ is a solution of* (6).

Now let $\mathfrak{D}(\boldsymbol{G})$ be the domain of the infinitesimal generator $\boldsymbol{G}$ of $\boldsymbol{T}_t$ in Hille-Yosida sense. Then we have the following

Theorem 3. *If $f\in\mathfrak{D}(\mathfrak{G})\cap \boldsymbol{C}^*(S)^+$, then $\hat{f}\in\mathfrak{D}(\boldsymbol{G})$ and*

$$\boldsymbol{G}\hat{f}=\langle f\mid c(f)\rangle, \tag{9}$$

where

$$c(f)=\mathfrak{G}f+k(\cdot)(F[\cdot\,;f]-f).$$

The next two theorems are the direct consequences of the above theorem.

Theorem 4. *If $f\in\mathfrak{D}(\mathfrak{G})\cap \boldsymbol{C}^*(S)^+$, then $u_t(x)=\boldsymbol{T}_t\hat{f}(x)$, $x\in S$ is in $\mathfrak{D}(\mathfrak{G})$ and*

$$\frac{\partial u_t}{\partial t}=\mathfrak{G}u_t+k(\cdot)(F[\cdot\,;u_t]-u_t),\text{[7]} \tag{10}$$

$$\|u_t-f\|\longrightarrow 0,\qquad (t\longrightarrow 0).$$

Theorem 5. *Put $A_t(\boldsymbol{x}, f)=\boldsymbol{T}_t\hat{f}(\boldsymbol{x})$, $\boldsymbol{x}\in\boldsymbol{S}-\{\Delta\}$, $f\in\mathfrak{D}^+$.*[8] *If $f\in\mathfrak{D}(\mathfrak{G})\cap\mathfrak{D}^+$, then $A_t(\boldsymbol{x}, f)$ is differentiable in t, $D_{c(f)}A_t(\boldsymbol{x}, f)$ exists and we have*

$$\frac{\partial A_t}{\partial t}=D_{c(f)}A_t, \tag{11}$$

$$A_{0+}(\boldsymbol{x}, f)=\hat{f}(\boldsymbol{x})$$

In [3] we have called (10) and (11) the backward and the forward equation respectively. Now we shall see that semi-group $\boldsymbol{T}_t$ of Theorem 2 is determined completely by (10) and (11). Namely we have

Theorem 6. *Let T_t' be a contruction semi-group on $\boldsymbol{B}_0(\boldsymbol{S})$ such that $\|T_t'f-f\|_S\longrightarrow 0$ when $t\longrightarrow 0$ for every $f\in \boldsymbol{C}_0(\boldsymbol{S})$.*

(i) *If $u_t'(x)=\boldsymbol{T}_t'\hat{f}(x)$, $x\in S$ satisfies* (10), *then $\boldsymbol{T}_t'=\boldsymbol{T}_t$.*

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7) $\frac{\partial u_t}{\partial t}$ is the strong derivative.

8) $\mathfrak{D}^+=\{f\in C(S), 0<f<1\}$. For the definition of $Dc(f)At$, we refer to [3].

(ii) *If* $A'_t(\boldsymbol{x}, f)=\boldsymbol{T}'_t\hat{f}(\boldsymbol{x})$, $\boldsymbol{x}\in\boldsymbol{S}-\{\Delta\}$, $f\in\mathfrak{D}^+$ *satisfies* (11), *then* $\boldsymbol{T}'_t=\boldsymbol{T}_t$.

The proof of (i) is reduced to the uniqueness of the solution of (10) for a given $f\in\boldsymbol{C}^*(S)^+$, while the proof of (ii) is based on the following lemma concerning the uniqueness of the solution of (11).

Lemma 2. *Let* $A_t(f)$ *be a (real-valued) function defined on* $t\in[0,\infty)$ *and* $f\in\mathfrak{D}^+$ *which satisfies the following conditions*:

(a) *If* $f\in\mathfrak{D}^+\cap\mathfrak{D}(\mathfrak{G})$, *then* $A_t(f)$ *is continuously differentiable in* t *and* $D_{c(f)}A_t(f)$ *exists.*

(b) *For every* $0<r<1$, *we have*

$$|A_t(f)-A_t(g)|\leqq C_r\,\|f-g\|,$$

provided that $f, g\in\mathfrak{D}^+$, $0<f, g\leqq r$, *and* $t\geqq 0$, *where* C_r *is a positive constant.*

(c) *For every* $0<r<1$ *we have*

$$|D_{c(f)}A_t(f)-D_{c(g)}A_t(g)| \leqq a_r\,\|c(f)\|\,\|f-g\|+b_r\|c(f)-c(g)\|$$

for every t, *provided that* $f, g\in\mathfrak{D}^+\cap\mathfrak{D}(\mathfrak{G})$, $0<f, g<1$, *where* a_r *and* b_r *are positive constants.*

If A_t *satisfies*

$$\frac{\partial A_t}{\partial t}(f)=D_{c(f)}A_t(f), \qquad A_{0+}(f)=0, \tag{12}$$

at every $f\in\mathfrak{D}^+\cap\mathfrak{D}(\mathfrak{G})$, *then we have* $A_t(f)\equiv 0$.

Now, we can prove that the semi-group $\boldsymbol{T}_t$ of Theorem 2 coincides with the semi-group $\tilde{\boldsymbol{T}}_t$ constructed in [5], where $\tilde{\boldsymbol{T}}_t$ is given by

$$\tilde{\boldsymbol{T}}_t f(\boldsymbol{x})=\sum_{n=0}^{\infty} T_t^n f(\boldsymbol{x}), \tag{13}$$

T_t^n is defined by (3.1) and (3.2) of [5], and it gives the minimal solution of M-equation. We have proved in [5] that $\tilde{\boldsymbol{T}}_t$ is a branching semi-group (the proof is essentially based on Theorem 1 of [2]) and so it satisfies the S-equation (6). Therefore in order to conclude that $\tilde{\boldsymbol{T}}_t=\boldsymbol{T}_t$, we are able to use the uniqueness of the solution of (6).

The above proof depends explicitely on the branching property of $\tilde{\boldsymbol{T}}_t$. In the following we give another proof based on the uniqueness of the solution of the forward equation (11) not depending on the branching property of $\tilde{\boldsymbol{T}}_t$. Define a kernel $\mu(\boldsymbol{x}, d\boldsymbol{y})$ on $(\boldsymbol{S}-\{\Delta\})\times(\boldsymbol{S}-\{\Delta\})$ by

$$\int_{\boldsymbol{S}}\hat{f}(\boldsymbol{y})\mu(\boldsymbol{x}, d\boldsymbol{y})=\langle f\mid kF[\cdot\,; f]\rangle(\boldsymbol{x}),$$

(cf. Lemma 2.2 of [3]), then the operator $\Psi(t)$ defined by (2.12) in [5] is given by

$$\Psi(t)f(\boldsymbol{x})=\int_0^t dr\int_{S-\{\Delta\}}\varphi(r,\boldsymbol{x},d\boldsymbol{y})f(\boldsymbol{y}),\qquad f\in\boldsymbol{C}_0(\boldsymbol{S}),\tag{14}$$

where

$$\varphi(t,\boldsymbol{x},d\boldsymbol{y})=\int_{S-\{\Delta\}}T^0(t,\boldsymbol{x},d\boldsymbol{z})\mu(\boldsymbol{z},d\boldsymbol{y}).$$

Now put[9]

$$\varphi^*(t,\boldsymbol{x},d\boldsymbol{y})=\int_{S-\{\Delta\}}\mu(\boldsymbol{x},d\boldsymbol{z})\,T^0(t,\boldsymbol{z},d\boldsymbol{y}),\tag{15}$$

then clearly we have

$$\int_{S-\{\Delta\}}\varphi(s,\boldsymbol{x},d\boldsymbol{z})T^0(t-s,\boldsymbol{z},d\boldsymbol{y})=\int_{S-\{\Delta\}}T^0(s,\boldsymbol{x},d\boldsymbol{z})\varphi^*(t-s,\boldsymbol{z},d\boldsymbol{y}).$$

This relation, (13) and (15) permit us to have

$$\tilde{\boldsymbol{T}}_t\hat{f}(\boldsymbol{x})=\widehat{T_t^0 f}(\boldsymbol{x})+\int_0^t ds\,\tilde{\boldsymbol{T}}_s(\langle T_{t-s}^0 f\mid kF[\cdot,T_{t-s}^0 f]\rangle)(\boldsymbol{x}).\tag{16}$$

From this formula we have by some simple calculations that $\tilde{A}_t(\boldsymbol{x},f)\equiv\tilde{\boldsymbol{T}}_t\hat{f}(\boldsymbol{x})$ satisfies (11) and so by Theorem 6 (ii) we have

$$\tilde{\boldsymbol{T}}_t=\boldsymbol{T}_t.$$

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9) The following discussion is similar to the arguments usually given in the proof that every minimal Markov chain satisfies forward differential equation. Cf. [1].