221. On Branching Semi-Groups. I

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In the previous papers we have given a definition of branching Markov processes (abbreviated as B.M.P.) [1], discussed some fundamental equations of B.M.P. [2], and constructed B.M.P. in a probabilistic way [3], [4]. This paper is a continuation of these papers and is devoted to an analytic construction of B.M.P. We shall treat this problem, however, in a little wider setup which may permit us to deal with the not necessarily positive branching semi-groups. (c.f. [5]).

1. Definition of branching semi-groups. Let S be a compact Hausdorff space with countable base, S^n be the n-fold symmetric product of S ($S^0=\{\partial\}$, an isolated point), and $\boldsymbol{S}=\bigcup_{n=0}^{\infty} S^n \cup \{\Delta\}$ be the one-point compactification of $\bigcup_{n=0}^{\infty} S^n$.[1] We denote by $\boldsymbol{C}(\boldsymbol{S})$ (resp. $\boldsymbol{C}(S)$ and $\boldsymbol{C}(S^n)$) the space of bounded continuous functions on $\boldsymbol{S}$ (resp. on S and S^n). $\boldsymbol{B}(\boldsymbol{S})$ is the space of bounded Borel measurable functions on $\boldsymbol{S}$. $\boldsymbol{C}_0(\boldsymbol{S})$ (resp. $\boldsymbol{B}_0(\boldsymbol{S})$) is the subspace of $\boldsymbol{C}(\boldsymbol{S})$ (resp. $\boldsymbol{B}(\boldsymbol{S})$) the elements of which vanish at infinity Δ.

Definition 1.1. A contraction[2] semi-group $\{T_t; t\geqq 0\}$ of linear operators on $\boldsymbol{C}(\boldsymbol{S})$ (or $\boldsymbol{B}(\boldsymbol{S})$) is said to be a *branching semi-group* (or *of branching property*), if it satisfies

$$(1.1)\qquad T_t\hat{f}(\boldsymbol{x})=(\widehat{T_t\hat{f}\,|_S})(\boldsymbol{x}), \qquad \boldsymbol{x}\in\boldsymbol{S},^{3)}$$

for any $f\in\bar{C}^*(S)$ (or $\bar{B}^*(S)$).[4]

Remark. Let $\boldsymbol{B}$ be a Banach space or Hilbert space, $\boldsymbol{B}^n=\boldsymbol{B}\otimes\boldsymbol{B}\otimes\cdots\otimes\boldsymbol{B}$ be the n-fold symmetric direct product of $\boldsymbol{B}$, and $\mathscr{B}=\sum_{n=0}^{\infty}\oplus\boldsymbol{B}^n$ ($B^0=\{\text{constants}\}$) be the direct sum of $\boldsymbol{B}^n$. Then the notion of branching semi-groups can be extended to a semi-group of linear operators on $\mathscr{B}$.

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1) For precise definition of $\boldsymbol{S}$, we refer to [1].

2) i.e. $\|T_t\|\leqq 1$. We do not assume positivity of T_t.

3) For $f\in\bar{\boldsymbol{B}}^*(S)$, we put $\hat{f}(\boldsymbol{x})=\prod_{j=1}^{n} f(x_j)$ if $x\in S^n$, $=0$ if $\boldsymbol{x}=\Delta$, and $=1$ if $\boldsymbol{x}=\partial$.

4) $\boldsymbol{C}^*(S)$ ($\boldsymbol{B}^*(S)$) $=\{f; f$ is bounded continuous (resp. Borel measurable) with $\|f\|<1\}$. $\bar{\boldsymbol{C}}^*(S)$ ($\bar{\boldsymbol{B}}^*(S)$) is the uniform closure of $\boldsymbol{C}^*(S)$ ($\boldsymbol{B}^*(S)$).

2. *M*-equation. We assume that we are given a system $\{T_t^0, K(t), \mu_n\}$ of quantities satisfying the following conditions:

[P. 1] There is a kernel[5] $T^0(t, x, dy)$ on $[0, \infty)\times S\times S$, and if we put

(2.1) $$T_t^0 f(x)=\int_S T^0(t, x, dy)f(y), \quad \text{for } x\in S \text{ and } f\in \boldsymbol{C}(S),$$

then T_t^0 is a strongly continuous contraction semi-group of linear operators on $\boldsymbol{C}(S)$.

[P. 2] There is a kernel[6] $K(x, dr, dy)$ on $S\times[0, \infty)\times S$, and if we put

(2.2) $$K(t)f(x)=\int_0^t\int_S K(x, dr, dy)f(y), \quad \text{for } x\in S \text{ and } f\in \boldsymbol{C}(S),$$

then $K(t)$ is a bounded linear operator on $\boldsymbol{C}(S)$. Moreover T_t^0 and $K(t)$ have the following relations; for any s and $t\geqq 0$,

(2.3) $$T_t^0 K(s)f(x)=K(t+s)f(x), \qquad x\in S,$$

and if we denote the total variations of $T^0(t, x, dy)$ and $K(t, dr, dy)$ by $|T^0|(t, x, dy)$ and $|K|(x, dr, dy)$, respectively, they satisfy

(2.4) $$|T^0|(t, x, S)-|T^0|(s, x, S)+\int_s^t |K|(x, dr, S)\leqq 0, \text{ for } s\leqq t, \text{ and}$$

(2.5) $$\lim_{t\downarrow 0}\sup_{x\in S}|K|(x, [0, t], S)=0.$$

[P. 3] There are kernels $\mu_n(x, dy)$ on $S\times S^n$, $n=0,1,2,\cdots,+\infty$,[7] and if we put

(2.6) $$\mu_n[f](x)=\int_{S^n}\mu_n(x, d\boldsymbol{y})f(\boldsymbol{y}), \quad x\in S \text{ and } f\in \boldsymbol{C}(S^n),$$

then μ_n is a bounded linear operator from $\boldsymbol{C}(S^n)$ to $\boldsymbol{C}(S)$ and satisfies

(2.7) $$\sum_{n=0}^{\infty}|\mu_n|(x, S^n)\leqq 1,$$

Where convergence is uniform in x.

By virtue of Lemma 2.2 of [2], we have

Lemma 2.1. *There exist unique kernels* $\boldsymbol{T}^0(t, \boldsymbol{x}, d\boldsymbol{y})$ *and* $\Psi(t, \boldsymbol{x}, d\boldsymbol{y})$ *on* $[0, \infty]\times\boldsymbol{S}\times\boldsymbol{S}$ *such that:*

1°) *If* $\boldsymbol{x}\in S^n$, $T^0(t, \boldsymbol{x}, .)$ *is concentrated on* S^n, *and if we put for* $f\in\boldsymbol{C}(S^n)$

(2.8) $$\boldsymbol{T}_t^0 f(x)=\int_{S^n}T^0(t, \boldsymbol{x}, d\boldsymbol{y})f(\boldsymbol{y}), \qquad \boldsymbol{x}\in S^n,$$

then $\boldsymbol{T}_t^0$ *is a strongly continuous semi-group on* $\boldsymbol{C}(S^n)$ *and satisfies*

$$\widehat{\boldsymbol{T}_t^0 f}(x)=\widehat{T_t^0 f}(\boldsymbol{x}), \quad \textit{for } f\in\boldsymbol{C}(S),\ \boldsymbol{x}\in S^n, n=1, 2, \cdots,$$

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5) For fixed $x\in S$ and $t\geqq 0$, $T^0(t, x,.)$ is a signed measure on S with bounded total variation, and for any fixed Borel set $B\subset S$, $T^0(., ., B)$ is Borel measurable on $[0, \infty)\times S$.

6) For fixed $x\in S$, $K(x, ., .)$ is a signed measure on $[0, \infty)\times S$ with bounded total variation, and for a fixed Borel set $B_1\times B_2$ of $[0, \infty)\times S$, $K(., B_1, B_2)$ is Borel measurable on S.

7) $S^0=\{\partial\}$ and $S^\infty=\{\Delta\}$.

(2.9) $$\boldsymbol{T}_t^0 f(\partial)=f(\partial), \quad \textit{for } f\in \boldsymbol{C}(S^0),$$
$$\boldsymbol{T}_t^0 f(\Delta)=f(\Delta), \quad \textit{for } f\in \boldsymbol{C}(\{\Delta\}),$$

and

(2.10) $$|\boldsymbol{T}_t^0|(t,\boldsymbol{x},S^n)\leqq 1, \quad \textit{if } \boldsymbol{x}\in S^n.$$

2°) *If we put*

(2.11) $$\Psi(t)f(\boldsymbol{x})=\int_S \Psi(t,\boldsymbol{x},d\boldsymbol{y})f(\boldsymbol{y}), \quad f\in \boldsymbol{C}_0(\boldsymbol{S}), \quad \boldsymbol{x}\in S^n,$$

then $\Psi(t)$ *is a bounded linear operator from* $\boldsymbol{C}_0(\boldsymbol{S})$ *to* $\boldsymbol{C}(S^n)$ *and satisfies*

(2.12) $$\Psi(t)f(\boldsymbol{x})=\int_0^t <\boldsymbol{T}_r^0\hat{f}\,|_S \Big| \int_S K(\cdot,dr,dy)\sum_{n=0}^{\infty}\mu_n[\hat{f}](y)>(\boldsymbol{x}),^{8)}\ \boldsymbol{x}\in \boldsymbol{S},$$

where $f\in \boldsymbol{C}^*(S)$.

3°) *Let* $|\Psi|(t,\boldsymbol{x},d\boldsymbol{y})$ *be the total variation of* Ψ, *then* $|\Psi|(\cdot,\boldsymbol{x},\boldsymbol{S})$ *is of bounded variation as a function of* t, *and it holds that*

(2.13) $$|\Psi|(t,\boldsymbol{x},\boldsymbol{S})\leqq 1-|\boldsymbol{T}^0|(t,\boldsymbol{x},\boldsymbol{S}), \quad \boldsymbol{x}\in \boldsymbol{S},$$

and

(2.14) $$\lim_{t\downarrow 0}\sup_{\boldsymbol{x}\in S^n}|\Psi|(t,\boldsymbol{x},\boldsymbol{S})=0, \quad n\geqq 0.$$

4°) T_t^0 *and* $\Psi(t)$ *are related, for* $f\in C_0(\boldsymbol{S})$, *as*

(2.15) $$\Psi(t)f(\boldsymbol{x})=\Psi(s)f(\boldsymbol{x})+T_t^0\Psi(t-s)f(\boldsymbol{x}), \quad \boldsymbol{x}\in \boldsymbol{S},\ s\leqq t.$$

Definition 2.1. Let $\boldsymbol{T}_t^0$ and Ψ be those given in Lemma 2.1. Consider an equation on $\boldsymbol{S}$, for $f\in \boldsymbol{B}_0(\boldsymbol{S})$,

(2.16) $$u_t(\boldsymbol{x})=T_t^0 f(\boldsymbol{x})+\int_0^t\int_S \Psi(dr,\boldsymbol{x},d\boldsymbol{y})u_{t-r}(\boldsymbol{y}), \quad \boldsymbol{x}\in \boldsymbol{S},$$

and we call it *M-equation* corresponding to $\{T_t^0, K, \mu_n\}$. If $u_t(\boldsymbol{x})$ satisfies *M*-equation for $f\in \boldsymbol{C}_0(\boldsymbol{S})$ then

(2.17) $$\lim_{t\downarrow 0}u_t(\boldsymbol{x})=f(\boldsymbol{x}), \quad \boldsymbol{x}\in \boldsymbol{S},$$

and u_t is called *the solution of M-equation with the initial value f.*

3. The minimal solution of *M*-equation. Now we construct a solution of *M*-equation. The procedure adopted by Moyal in [6] is applicable for our case. Namely, if we put for $\boldsymbol{x}\in \boldsymbol{S}, t\geqq 0$, and $f\in \boldsymbol{B}_0(\boldsymbol{S})$

(3.1) $$\begin{cases}\Psi^{(0)}(t)f(\boldsymbol{x})=f(\boldsymbol{x}),\ \Psi^{(1)}(t)f(\boldsymbol{x})=\Psi(t)f(\boldsymbol{x}), \text{ and}\\ \Psi^{(n)}(t)f(\boldsymbol{x})=\int_0^t \Psi^{(n-1)}(dr)\Psi(t-r)f(\boldsymbol{x}), \quad \text{for} \quad n\geqq 2,^{9)}\end{cases}$$

and put

(3.2) $$\begin{cases}\boldsymbol{T}_t^{(0)}f(\boldsymbol{x})=\boldsymbol{T}_t^0 f(\boldsymbol{x}),\\ \boldsymbol{T}_t^{(n)}f(\boldsymbol{x})=\int_0^t \Psi^{(n)}(dr)\boldsymbol{T}_{t-r}^{(0)}f(\boldsymbol{x}), \quad \text{for} \quad n\geqq 1,\end{cases}$$

then we have

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8) For $f\in \boldsymbol{B}^*(S)$ and $g\in \boldsymbol{B}(S)$, we put $\langle f/g\rangle(\boldsymbol{x})=\sum_{k=1}^{n} g(x_k)\prod_{j\neq k} f(x_j)$ if $\boldsymbol{x}\in S^n$, and $=0$ if $\boldsymbol{x}=\partial$ or Δ.

9) For convenience, we write $\Psi(dt)f(x)$ rather than $d\Psi(t)f(x)$.

Lemma 3.1. $\boldsymbol{T}_t^{(n)}$ and $\Psi^{(n)}(t)$ *satisfy for* $\boldsymbol{x}\in\boldsymbol{S}$ *and* $f\in\boldsymbol{B}_0(\boldsymbol{S})$

(3.3) $$\Psi^{(n)}(t)f(\boldsymbol{x})=\int_0^t\Psi^{(n-k)}(dr)\Psi^{(k)}(t-r)f(\boldsymbol{x}),\quad for\quad k=0,1,2,\cdots,n,$$

(3.4) $$\boldsymbol{T}_t^{(n)}f(\boldsymbol{x})=\int_0^t\Psi^{(n-k)}T_{t-r}^{(k)}f(\boldsymbol{x}),\quad for\quad k=0,1,2,\cdots,n,$$

(3.5) $$T_r^{(0)}T_{t-r}^{(n)}f(\boldsymbol{x})=\int_r^t\Psi(ds)\boldsymbol{T}_{t-s}^{(n-1)}f(\boldsymbol{x}),$$

(3.6) $$\Psi^{(n)}(t)f(\boldsymbol{x})=\Psi^{(n)}(s)f(\boldsymbol{x})+\sum_{j=1}^{n}\boldsymbol{T}_s^{(n-j)}\Psi^{(j)}(t-s)f(\boldsymbol{x}),\qquad n\geqq 1.$$

Lemma 3.2. *For any* $f\in\boldsymbol{B}_0(\boldsymbol{S})$,

(3.7) $$\sum_{n=0}^{N}\boldsymbol{T}_t^{(n)}f(\boldsymbol{x})$$

converges absolutely when N *tends to infinity.*

Lemma 3.3. *There exists a kernel* $\boldsymbol{T}(t,\boldsymbol{x},d\boldsymbol{y})$ *on* $[0,\infty)\times\boldsymbol{S}\times\boldsymbol{S}$ *such that*

(*i*) *if we put*

(3.8) $$\boldsymbol{T}_tf(\boldsymbol{x})=\int_S\boldsymbol{T}(t,\boldsymbol{x},d\boldsymbol{y})f(\boldsymbol{y}),\quad for\quad f\in\boldsymbol{B}_0(\boldsymbol{S}),$$

then it holds that

(3.9) $$\boldsymbol{T}_tf(\boldsymbol{x})=\sum_{n=0}^{\infty}\boldsymbol{T}_t^{(n)}f(\boldsymbol{x}),\quad and$$

(3.10) $$\|\boldsymbol{T}_t\|\leqq 1.$$

(*ii*) $\boldsymbol{T}(t,\boldsymbol{x},d\boldsymbol{y})$ *satisfies Chapman-Kolmogorov's equation.*

(*iii*) *If we put* $u_t(\boldsymbol{x})=T_tf(\boldsymbol{x})$ *for* $f\in C_0(\boldsymbol{S})$, *then* u_t *is a solution of M-equation with the initial value* f *and it satisfies*

(3.11) $$\lim_{t\downarrow 0}\|\boldsymbol{T}_tf-f\|=0.$$

Lemma 3.4. *The semi-group* $\boldsymbol{T}_t$ *defined by* (3.8) *is a branching semi-group.*

Proof of this lemma heavily leans upon the results of [1]. Combining the above lemmas, we have

Theorem 3.1. *Let* $\{T_t^0,K(t),\mu_n\}$ *satisfying* [P. 1], [P. 2], *and* [P. 3] *be given, then there exists a kernel* $\boldsymbol{T}(t,\boldsymbol{x},d\boldsymbol{y})$ *on* $[0,\infty)\times\boldsymbol{S}\times\boldsymbol{S}$, *and if we define* $\boldsymbol{T}_t$ *by* (3.8), *it satisfies*:

(*i*) $\boldsymbol{T}_t$ *is a branching semi-group on* $\boldsymbol{B}_0(\boldsymbol{S})$ *satisfying* (3.9) *and* (3.10).

(*ii*) $\boldsymbol{T}_t$ *is strongly continuous at* $t=0$, *for* $f\in\boldsymbol{C}_0(\boldsymbol{S})$.

(*iii*) $u_t(\boldsymbol{x})=\boldsymbol{T}_tf(\boldsymbol{x})$ *is a solution of M-equation corresponding to* $\{T_t^0,K(t),\mu_n\}$ *with the initial value* f, *provided that* $f\in\boldsymbol{C}_0(\boldsymbol{S})$.

(*iv*) *If* T_t^0, $K(t)$ *and* μ_n *are non-negative, then* $\boldsymbol{T}_t$ *is also non-negative.*

(*v*) *If, for any bounded Borel measurable function* $v(t,x)$ *on* $[0,\infty)\times S$, *we have*

(3.12) $$\int_0^t\int_S K(\cdot,dr,dy)v(t-r,y)\in\boldsymbol{C}(\boldsymbol{S}),$$

then, $\boldsymbol{T}_t$ is a strongly continuous semi-group on $\boldsymbol{C}_0(\boldsymbol{S})$.

Now we give a condition under which the solution of M-equation becomes unique.

Proposition 3.1. *If Ψ satisfies, for any $T>0$,*

$$\sup_{t\leq T}\sup_{x\in S^n}|\Psi|(t, \boldsymbol{x}, \boldsymbol{S})<1, \quad \textit{for } n=1, 2, \cdots, \tag{3.13}$$

then the bounded solution $u(t, \boldsymbol{x})$ of M-equation with

$$\lim_{\boldsymbol{x}\to\Delta}\sup_{t\leq T}|u(t, \boldsymbol{x})|=0 \tag{3.14}$$

is unique.

Corollary. *Assume that* (3.13) *is satisfied. If there exists a branching semi-group T_t satisfying M-equation and if it satisfies for $f\in \boldsymbol{C}^*(S)$ and $T>0$,*

$$\sup_{t\leq T}\sup_{x\in S}|T_t\hat{f}(x)|<1,$$

then T_t coincides with the semigroup $\boldsymbol{T}_t$ obtained in Theorem 2.1.

Remark. If $\inf_{t\leq T}\inf_{x\in S}|T^0|(t, x, S)>0$, (3.13) is satisfied.

As a concequence of Theorem 3.1, we are able to construct a branching Markov process corresponding to a fundamental system $\{T_t^0, K, q_n, \pi_n\}$.[10]

Remark. Put $\sigma_\infty(t, \boldsymbol{x})=\lim_{n\to\infty}\Psi^{(n)}(t, \boldsymbol{x}, \boldsymbol{S})$. Moyal proved in [6] that the bounded solution of M-equation corresponding to a fundamental system is unique if and only if $\sigma_\infty(\infty, \boldsymbol{x})=0$, for every $\boldsymbol{x}\in \boldsymbol{S}$. The probabilistic meaning of this condition is obvious. For, $\sigma_\infty(\infty, \boldsymbol{x})=P_{\boldsymbol{x}}[0\leqq\tau_\infty<\infty]$. If $P_{\boldsymbol{x}}[0\leqq\tau_\infty<\infty]>0$, there may appear many solutions of M-equation which correspond to "branching Markov processes with instantaneous return" from some "boundary" of B.M.P. They may be constructed in the same way as [3] by giving some "instantaneous distribution", but their semi-group have no longer the branching property.

4. S-equation and its relation to $\boldsymbol{M}$-equation. Given a system $\{T_t^0, K(t), \mu_n\}$ satisfying [P. 1], [P. 2], and [P. 3], we put, for $f\in \bar{B}^*(S)$

$$F[f](x)=\sum_{n=0}^{\infty}\mu_n[\hat{f}](x), \qquad x\in S. \tag{4.1}$$

Definition 4.1 Consider an equation on S, for $f\in \bar{\boldsymbol{B}}^*(S)$,

$$u_t(x)=T_t^0f(x)+\int_0^t\int_S K(x, dr, dy)F[u_{t-r}](y), \qquad x\in S, \tag{4.2}$$

and we call it *S-equation* corresponding to $\{T_t^0, K(t), \mu_n\}$ and if u_t satisfies S-equation for $f\in \bar{\boldsymbol{C}}^*(S)$ then

$$\lim_{t\downarrow 0}u_t(x)=f(x),$$

and it is called *a solution of S-equation with the initial value f.*

10) For the definition of fundamental systems we refer to [2].

At first, we have

Proposition 4.1. *If T_t is a branching semi-group, and if $u(t, \boldsymbol{x}) = T_t\hat{f}(\boldsymbol{x})$ satisfies M-equation for $f \in \bar{C}^*(S)$, then*

$$u_t(x) = (T_t\hat{f})|_S(x) \tag{4.3}$$

is a solution of S-equation.

Proof is easily performed.

Next we note that a converse of this proposition is valid.

Lemma 4.1. *If $u_t(x)$ satisfies S-equation*

$$u_t(x) = T_t^0 f(x) + \int_0^t \int_S K(x, dr, dy) F[u_{t-r}](y), \qquad x \in S,$$

where $f \in \bar{B}^*(S)$, then it holds that

$$\widehat{T_S^0 u_{t-s}}(\boldsymbol{x}) = \widehat{T_t^0 f(\boldsymbol{x})} + \int_s^t < T_r^0 u_{t-r} \Big| \int_S K(\cdot, dr, dy) F[u_{t-r}](y) > (\boldsymbol{x}), \quad \boldsymbol{x} \in \boldsymbol{S}. \tag{4.4}$$

By virtue of this lemma, we have

Theorem 4.1. *If $u_t(x)$ is a solution of S-equation with the initial value $f \in \boldsymbol{C}^*(S)$, then $\hat{u}_t(\boldsymbol{x})$ is a solution of M-equation with the initial value $\hat{f}$.*

Corollary. *If for any $T > 0$,*

$$\inf_{t \leq T} \inf_{x \in S} | T^0 | (t, x, S) > 0,$$

then the solution $u(t, x)$ of S-equation for $f \in \boldsymbol{C}^*(S)$ with $\sup_{t \leq T} || u(t, .) || < 1$ is unique.

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