244. *Representations of Linear Continuous Functionals on the Space* C(X, Y) *of Continuous Functions from Compact X into Locally Convex* Y*)

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Let R be the space of reals and Y, Z, W be real Banach spaces. Denote by U the space of all bilinear continuous operators u from the space $Y\times Z$ into W. Norms of elements in the spaces Y, Y', Z, W, U will be denoted by $|\quad|$.

A nonempty family of sets V of an abstract space X is called a prering if the following conditions are satisfied: (a) if $A_1, A_2 \in V$, then $A_1 \cap A_2 \in V$, (b) if $A_1, A_2 \in V$, then there exist disjoint sets $B_1, \cdots, B_k \in V$ such that $A_1 \backslash A_2 = B_1 \cup \cdots \cup B_k$. A function μ from a prering V into a Banach space Z is called a volume if it satisfies the following condition: for every countable family of disjoint sets $A_t \in V (t \in T)$ such that (c) $A = \cup_T A_t \in V$, we have $\mu(A) = \sum_T \mu(A_t)$, where the last sum is convergent absolutely and $|\mu|(A) = \sup\{\sum |\mu(A_t)|\} < \infty$ for any $A \in V$, where the supremum is taken over all possible decompositions of the set A into the form (c).

A volume is called positive if it takes on only nonnegative values. If μ is a volume, then its variation $|\mu|$ is a positive volume.

The triple (X, V, v), where v is a fixed positive volume will be called a volume space.

In [1] has been developed the theory of the integral of the form $\int u(f, d\mu)$ defined for $f \in L(|\mu|, Y)$ and $u \in U$, and the theory of the space $L(|\mu|, Y)$ of Lebesgue-Bochner summable functions. For the case of locally compact Hausdorff spaces this integral essentially generates the same operator on the space of summable functions as the integral developed in a different way by Bourbaki [4], Ch. VI, p. 48–49.

The main advantage of the construction of the Lebesgue-Bochner-Stieltjes integration presented in [1] is that one may integrate with respect to any volume μ defined on a prering V without extending it first to a measure.

In this paper by the integral $\int u(f, d\mu)$ we shall understand the integral developed in [1] considered on the space $L(|\mu|, Y)$.

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The rest of the terminology and notation is the same as in [2] where has been presented an approach to the theory of integration generated by positive linear functionals defined on the space C_0 of real-valued continuous functions with compact support defined on a locally compact space X.

§1. Linear continuous functionals on the space $C(X, Y)$ for compact X and a Banach space Y. Let X be a compact Hausdorff space and Y be a Banach space.

Let $C(X, Y)$ denote the Banach space of all continuous functions on X to Y. The norm in this space is defined as usual.

Let $M(V, Y)$ denote the space of all volumes μ on V to Y such that their variation $|\mu|$ is regular on the prering V. The space $M(V, Y)$ is linear.

If V is the Baire prering, then, according to Th. 5, §4 [13] (cf. also [11]), every positive volume on V is regular and therefore the variation $|\mu|$ of any volume μ is also regular.

The norm in the space $M(V, Y)$ is defined by $\|\mu\|=|\mu|(X)$. In the following theorem u denotes the operator $u(y, y')=y'(y)$ for $y \in Y, y' \in Y'$.

Theorem 1. (*cf.* [2], [8]) *Let X be a compact Hausdorff space and V its Baire prering. Let $C=C(X, Y)$ and $M=M(V, Y')$. To every linear continuous functional h on the space C corresponds a unique volume $\mu \in M$ such that $h(f)=\int u(f, d\mu)$ for all $f \in C$. This correspondence establishes an isomorphism and isometry of the space C' and M.*

The proof makes use of the results of [2] and some simple properties of the Baire prering.

§2. Extensions of vector-valued volumes from the Baire prering onto the Borel ring. Let B be the Borel ring and V be the Baire prering of X. Consider the mapping $\bar{\mu}=T\mu$ of the space $M(B, Y')$ into the space $M(V, Y')$ defined by the formula $\bar{\mu}(A)=\mu(A)$ for $A \in V$.

Theorem 2. *The mapping T establishes an isometry and isomorphism between the spaces $M(B, Y')$ and $M(V, Y')$. If $\bar{\mu}=T\mu$ then $|\bar{\mu}| \subset |\mu|$, where $|\mu|$ denotes the variation of μ.*

Let $N(v, Y)$ denote the set of all functions f from X into Y equal to zero v-almost everywhere.

Denote by $L_V(Y)$ the quotient space $L(v, Y)/N(v, Y)$. For an element $f \in L_V(Y)$ define $\int u(f, d\mu)=\int u(g, d\mu)$ and $\|f\|_V=\|g\|_V$, where the function g belongs to the class f and $v=|\mu|$.

Notice that this definition is correct, that is it does not depend on the choice of the representative $g \in f$.

Assume that $\bar{\mu}=T\mu$, $\mu \in M(B, Y')$. Put $v=|\mu|$ and $w=|\bar{\mu}|$. Let i denote a mapping of the space $L_V(Y)$ into the space $L_W(Y)$ defined as follows: for any two classes $f \in L_V(Y)$ and $g \in L_W(Y)$ we have $f=i(g)$ if and only if the classes f, g have at least one common element. Using the fact that in every class there exists a representative belonging to the second Baire class $C_2(Y)$ it is easy to prove that this mapping is well defined. For more details see [2].

Theorem 3. *The mapping i establishes an isometric isomorphism of the space $L_V(Y)$ with the space $L_W(Y)$ preserving the integral, that is the map i is linear and onto, $\|f\|_V = \|if\|_W$ and $\int u(f, d\mu)=\int u(if, d\bar{\mu})$ for all $f \in L_V(Y)$. (Cf. Th. 4, [2]).*

Using Theorems 2 and 3 we easily get Singer's theorem [10] (Cf. also [7]):

Theorem 4. *To every linear continuous functional h on the space $C=C(X, Y)$ corresponds a unique vector measure $\mu \in M=M(B, Y')$ such that*

$$h(f)=\int u(f, d\mu) \quad \text{for all} \quad f \in C.$$

This correspondence establishes an isometry and isomorphism between the spaces C' and M.

§ 3. Weak convergence in the space C(X, Y). Let the Banach space Y be such that for every $\mu \in M(V, Y')$ there exists a function g weakly summable on every set $A \in V$ and such that $\mu(A) =\int_A g d|\mu|$ for $A \in V$, where $g(x) \in Y'$ and $|g(x)| \le 1$ $|\mu|$-almost everywhere.

The space Y satisfies the above conditions, for example, if either one of the spaces Y or Y' is separable [5], [6] or if the space Y is reflexive [9].

Theorem 5. *Let $f, f_n \in C=C(X, Y)$. Then the sequence f_n converges weakly to the function f in the space C if and only if the sequence is bounded and $y'f_n(x)$ converges to $y'f(x)$ for all $y' \in Y'$ and $x \in X$.*

The proof makes use of the following:

Lemma 1. *If fixed μ, g satisfy the above condition and u is any bilinear continuous functional on $Y \times Y'$ then for every $f \in L(|\mu|, Y)$ the function $u(f, g)$ belongs to $L(|\mu|, R)$ and $\int u(f, d\mu) =\int u(f, g)d|\mu|$.*

§ 4. Weak convergence of functionals on the space C(X, Y).

Theorem 6. *Let h, h_n be linear continuous functionals on $C=C(X, Y)$ and $\mu, \mu_n \in M(V, Y')$ corresponding volumes generated by them. The sequence $h_n(f)$ converges to the value $h(f)$ for every*

$f \in C$ if and only if the sequence μ_n is bounded and $\int f d\mu_n(\cdot, y) \to \int f\, d\mu(\cdot, y)$ for every $f \in C(X, R)$ and $y \in Y$.

This theorem follows immediately from the fact that the functions of the form $f = f_1 y_1 + \cdots + f_n y_n$, $f_i \in C(X, R)$, $y_i \in Y$ lie densely in the space $C(X, Y)$.

Theorem 7. *Let $\mu, \mu_n \in M(V, Z)$ and $u \in U$. If $\|\mu_n\| \le c$ for $n = 1, 2, \cdots$ and $\mu_n(A) \to \mu(A)$ for all $A \in V$ then $\|\mu\| \le c$, and*

$$\int u(f, d\mu_n) \to \int u(f, \mu)$$

for all $f \in C(X, Y)$.

The following corollary follows immediately from Theorems 6 and 7.

Corollary 1. *If $\mu, \mu_n \in M(V, Y')$ are such that $\mu_n(A, y) \to \mu(A, y)$ for all $A \in V$ and $y \in Y$ then the corresponding functionals $h_n(f)$ converge to the value $h(f)$ for every $f \in C(X, Y)$.*

§ 5. **Integration of locally-convex—space-valued functions with respect to some vector measures.** Let Y be a locally convex space (not necessarily complete or Hausdorff) and let P denote the family of all continuous seminorms on it. If $p \in P$ then the set $N_p = \{y \in Y : p(y) = 0\}$ is linear. Consider the quotient space Y/N_p with the norm defined by the formula $\|y + N_p\|_p = p(y)$. Denote the completion of the space Y by Y_p.

Let T_p be the natural mapping of the space Y into the space Y_p. We have $\|T_p y\|_p = p(y)$ for all $y \in Y$. Let T'_p be the conjugate mapping of the space Y'_p into the space Y'. That is we have

$$u(y, T'_p z) = u_p(T_p y, z)$$

for all $y \in Y, z \in Y'_p$, where $u(y, y') = y'(y)$ for all $y' \in Y', y \in Y$, and $u_p(y, y') = y'(y)$ for all $y' \in Y'_p, y \in Y_p$.

Let X be as before, a compact Hausdorff space. Denote by $C(X, Y) = C$ the space of all continuous functions f from the set X into Y. In this space we introduce locally convex topology by means of the seminorms

$$p(f) = \sup \{p(f(x)) : x \in X\},\ p \in P.$$

Let V be the Baire prering of the space X. Consider the set of all measures of the form $\mu = T'_p \circ \mu_p$, that is $\mu(A) = T'_p(\mu_p(A))$ for all $A \in V$, where $\mu_p \in M(V, Y'_p)$. This set will be denoted by M_p. Put $M = U_{p \in P} M_p$.

Now take any $\mu \in M$. There exists a seminorm $p \in P$ such that $\mu \in M_p$. That is we have $\mu = T'_p \circ \mu_p$, $\mu_p \in M(V, Y'_p)$. Take any $f \in C$ and define

$$\int u(f, d\mu) = \int u_p(T_p \circ f, d\mu_p).$$

Theorem 8. (1) *The operator* $\int u(f, d\mu)$ *is well defined and represents a bilinear functional on the space* $C \times M$. (2) *If* $f, f_n \in C$ *and* $f_n \to f$ *in the topology of* C *then*

$$\int u(f_n, d\mu) \to \int u(f, d\mu) \text{ for every } \mu \in M.$$

§ 6. Representations of linear continuous functionals on the space C(X, Y) for compact Hausdorff X and locally convex Y.

Let X, Y, C, M be as before.

Theorem 9. (1) *To every linear continuous functional* h *on the space* $C = C(X, Y)$ *corresponds a unique vector valued measure* $\mu \in M = U_p M_p (p \in P)$, *where* $M_p = T'_p \circ M(V, Y'_p)$, *such that* $h(f) = \int u(f, d\mu)$ *for all* $f \in C$, *and conversely.* (2) *If* $|h(f)| \leq cp(f)$ *for all* $f \in C$ *then the corresponding measure* μ *is of the form* $\mu = T'_p \circ \mu_p$, $\mu_p \in M(V, Y'_p)$, *and the least constant* c *satisfying the above inequality can be found by the formula*

$$c_p = \| \mu_p \| = sup \; | \textstyle\sum_k u(y_k, \mu(A_k)) |,$$

where the supremum is taken over all finite systems of disjoint sets $A_k \in V$ *and all* $y_k \in Y$ *such that* $p(y_k) \leq 1$.

The above result represents a generalization of one of the results of Swong [12] who has found a similar representation by means of a Riemann-Stieltjes type integral for the case when Y is a complete locally convex Hausdorff space. He has proven that one may use to represent the functionals the measures of the form $\mu = T'_p \circ \mu_p$, where μ_p are weakly regular with finite semivariation defined on the Borel ring.

These results will appear in Mathematische Annalen.

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