8. *Algebraic Formulation of Propositional Calculi with General Detachment Rule*

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R. B. Angell [1] formulated a general detachment rule: α and $\varphi(c\alpha\beta)$ imply $\varphi(\beta)$, and further I. Thomas [7] considered on this general detachment rule.

On the other hand, in my notes ([3], [4]), I gave a fundamental idea of algebraic formulations of propositional calculi. This is as follows: Let $M=\langle X, 0, \{o_\alpha\}\rangle$ be an algebra consisting of a set X containing a zero element 0 and a family of operations $\{o_\alpha\}$ containing a binary operation $*$. On the operation $*$, there are common properties: 1) $x*y=0$ is equivalent to $x\leqslant y$, 2) $x=y$ is defined by $x*y=y*x=0$. This means that if $x\leqslant y, y\leqslant x$, then $x=y$.

As easily seen from [1], [7], the general detachment rule is formulated in the form of '$x*0=x$ for all $x\in X$' in the algebra M. Therefore, if we add this axiom to the algebra M, we obtain an algebraic formulation of propositional calculi with a general detachment rule.

In this Note, we shall consider such algebras M.

In our notes ([2], [5]), if an algebra $M=\langle X, 0, *\rangle$ satisfies

1) $(x*y)*(x*z)\leqslant z*y$,
2) $x*(x*y)\leqslant y$,
3) $x\leqslant x$,
4) $x\leqslant 0$ implies $x=0$,

then M is called a BCI-*algebra*.

In the BCI-algebra, we have $(x*y)*z=(x*z)*y$ (see Theorem 1 in [5]). Hence we have $(x*0)*x=(x*x)*0=0$ by 3), and further $x*(x*0)=0$ by 2). This shows $x*0=x$ for all $x\in X$.

Then we have the following

Theorem 1. *An algebra M is a BCI-algebra if and only if M satisfies*

5) $((x*y)*z)*(u*z)\leqslant(x*u)*y$,
6) $x*0=x$,
7) $x\leqslant 0$ *implies* $x=0$.

Proof. Put $z=0$ in 5), then

8) $(x*y)*u\leqslant(x*u)*y$.

Hence we have $(x*y)*u=(x*u)*y$. Next put $y=0$ in 5), then

9) $(x*z)*(u*z)\leqslant x*u$.

By 8) and 9), we have

10) $(x*z)*(x*u)\leqslant u*z$,

which is axiom 1). This implies that $\leqslant$ is the transitive relation.

Put $z=0$ in 10), then

$$x*(x*u)\leqslant u,$$

which means axiom 2. Let $u=0$ in the relation above, then we have $x*x\leqslant 0$. Hence 7) implies $x\leqslant x$. Hence we complete the proof.

Theorem 2. *An algebra M is a BCI-algebra if and only if M satisfies*

11) $(x*y)*(x*z)\leqslant z*y$,

12) $x*0=x$,

13) $x\leqslant 0$ implies $x=0$.

Proof. We shall only prove the 'if' part. Put $y=0$ in 11), then, by 12), we have

14) $x*(x*z)\leqslant z$,

which is axiom 2). Let $z=0$ in 14), then we have $x*x\leqslant 0$. Therefore 13) implies $x*x=0$. This means $x\leqslant x$. We complete the proof.

In our Notes ([2], [5]), we define a BCK-algebra as follows: If axiom 4) in the BCI-algebra M is replaced by

15) $0\leqslant x$ for all $x\in X$,

then M is called a BCK-*algebra*. Of course '$x*0=x$ for all $x\in X$' holds in the BCK-algebra.

As easily seen from the proof of Theorem 2, we have the following

Theorem 3. *An algebra M is a BCK-algebra if and only if M satisfies*

16) $(x*y)*(x*z)\leqslant z*y$,

17) $x*0=x$,

18) $0\leqslant x$.

As an example, we take up an axiom by C. A. Meredith [6].

Theorem 4. *An algebra M is a BCK-algebra if and only if M satisfies*

19) $((x*y)*z)*(x*u)*y)\leqslant u*(z*v)$,

20) $x*0=x$,

21) $0\leqslant x$.

Proof. It is sufficient to show that the conditions 19), 20), and 21) imply axioms 1), 2), 3).

Put $v=0$ in 19), then

22) $((x*y)*z)*((x*u)*y)\leqslant u*z$.

Let $y=0$ in 22), then we have

23) $(x*z)*(x*u)\leqslant u*z$,

which is axiom 1), i.e. 16). Hence by Theorem 3, we have axioms

2) and 3). Therefore we complete the proof.

Further, we shall take up a thesis $(x*y)*(x*(z*(u*y)))\leqslant z*u$ by C. A. Meredith [6].

Theorem 5. *An algebra M is a BCI-algebra if and only if M satisfies*

24) $(x*y)*(x*(z*(u*y)))\leqslant z*u$,

25) $x*0=x$,

26) $x\leqslant 0$ implies $x=0$.

Proof. Let $y=0$ in 24), then

27) $x*(x*(z*u))\leqslant z*u$.

Put $u=0$ in 27), then we have

28) $x*(x*z)\leqslant z$,

which is axiom 2). Put $z=0$ in 28), then $x*x\leqslant 0$. By 26), we have $x*x=0$. This means

29) $x\leqslant x$.

Let $u=y$ in 24), then, $z*(u*y)=z*0=z$, we have

$$(x*y)*(x*z)\leqslant z*y,$$

which is axiom 1).

Remark. If the condition 26) is replaced by '$0\leqslant x$ for all $x\in X'$', then we have a characterization of a BCK-algebra. To prove it, put $z=0$ in 24), then by $z*(u*y)=0*(u*y)=0$, we have $(x*y)*x=0$, which means $x*y\leqslant x$. This completes the proof.

An algebra M is called an *I-algebra*, if M satisfies

30) $(x*y)*(x*z)\leqslant z*y$,

31) $x\leqslant x*(y*x)$,

32) $x*y\leqslant x$.

33) $0\leqslant x$.

We give some characterizations of *I*-algebra.

Theorem 6. *An algebra M is an I-algebra if and only if the following relations hold in M*:

34) $(x*y)*(x*z)\leqslant z*y$,

35) $x*y\leqslant x*(z*x)$,

36) $x*0=x$,

37) $0\leqslant x$.

Proof. We shall give a proof of 'if' part. Let $y=0$ in 34), then $x\leqslant x*(z*x)$. Next $z=0$ in 34), then we have $x*y\leqslant x$ by 36) and 37). Therefore we complete the proof.

Theorem 7. *An algebra M is an I-algebra if and only if the following relations hold in M*:

38) $((x*y)*z)*(x*u)\leqslant(u*y)*(v*x)$,

39) $x*0=x$,

40) $0\leqslant x$.

Proof. Let $z=v=0$ in 38), then we have

41) $(x*y)*(x*u)\leqslant u*y$,

which is axiom 30). Put $u=0$ in 41), then

42) $(x*y)*x=0$,

which is axiom 32), and further we have $x\leqslant x$. Next put $y=z=0$ in 38), then

43) $x*(x*u)\leqslant u*(v*x)$.

Let $u=y*x$, $v=y$ in 43), then

$$x*(x*(y*x))\leqslant(y*x)*(y*x)=0,$$

hence $x\leqslant x*(y*x)$, which is axiom 31).

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