4. *A Note on the Projective Modules over Group Rings*

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Let R be a commutative ring with identity, and K its total quotient ring. Let π be a finite group of order n. We assume that no prime number dividing n is a unit in R. Then, if P denotes a finitely generated projective $R\pi$-module, Swan [1, Theorem 8.1] has shown that $K\otimes_R P$ is $K\pi$-free under the condition that R is a Dedekind ring of characteristic zero. In this note we deal with this theorem in weaker conditions on R. In the following we assume all modules over $R\pi, K\pi, \cdots$ are finitely generated unitary left modules.

Lemma 1. (Brauer, Nesbitt [3, Theorem 30.16]) Let M, N be $K\pi$-modules, where K is a splitting field of π. Let $\boldsymbol{M}, \boldsymbol{N}$ be corresponding matrix representations. Then M and N have the same composition factors if and only if the matrices $\boldsymbol{M}(x), \boldsymbol{N}(x)$ have the same characteristic roots for each $x\in\pi$.

Lemma 2. (Giorgiutti, Rim [2, Lemma 2.2]) Let Λ be an Artinian ring of which Cartan matrix is non-singular. Then two projective Λ-modules with the same composition factors are isomorphic.

Lemma 3. (Swan, Bass [2, Theorem 2]) Let R be a commutative local ring with the maximal ideal $\mathfrak{m}$, and K its total quotient ring. Let $\mathfrak{O}$ be an R-projective R-algebra finitely generated as an R-module, and we assume that $R/\mathfrak{m}$-algebra $\mathfrak{O}/m\mathfrak{O}$ has the non-singular Cartan matrix. Then for any projective $\mathfrak{O}$-modules P and P', $K\otimes P\cong K\otimes P'$ implies $P\cong P'$.

A ring K is called semi-local if K/N is Artinian, where N denotes the Jacobson radical of K. If K is commutative, it is equivalent to say that there exist only a finite number of maximal ideals.

Lemma 4. Let K be a commutative semi-local ring, and π a finite group. Let P be a projective $K\pi$-module which is $K\pi'$-free for any cyclic subgroup π' of π, and we assume that the rank of P over K is divisible by the order of π. Then P is $K\pi$-free.

Proof. Let N be the Jacobson radical of K. It is known by [2, Lemma 2.4] that P is $K\pi$-free if and only if $(K/N)\otimes_K P$ is $(K/N)\pi$-free. So we may assume that $K=K_1\oplus\cdots\oplus K_r$ is a direct sum of the fields. If $P=P_1\oplus\cdots\oplus P_r$ is the corresponding decomposition, every P_i is $K_i\pi$-projective and $K_i\pi'$-free. If we prove that P_i is $K_i\pi$-free, P is $K\pi$-free because P is K-free and so the ranks of P_i over K_i are the same for all i's. Therefore we reduce the lemma to the case that K is a field. But then by Noether-Deuring theorem [3, Theorem 29.7], P is $K\pi$-free if and only if $K'\otimes_K P$ is $K'\pi$-free where K' is any extension of K. If we put K' as a splitting field of π, the assertion follows from Lemmas 1 and 2.

Remark. [3, Lemma 78.2] states that a $K\pi$-module is $K\pi$-free if and only if it is $K\pi_p$-free for each p and each p-Sylow subgroup π_p, where K is an algebraic number field. But this is not true. Let π be a cyclic group of order 6 generated by σ. Let K be an algebraic number field containing all the 6-th roots of unity. Then $K\pi\cong\sum K_\zeta$, where the sum runs over all the 6-th roots ζ of unity, and σ acts on $K_\zeta\cong K$ as multiplying ζ. Let $M=K_1+K_1+K_\omega+K_{-\omega}+K_{-\omega^2}+K_{-\omega^2}$, where ω is a primitive 3-rd root of unity. Then M is free over Sylow groups but not $K\pi$-free.

Lemma 5. Let π be a cyclic group of order $n=p^d m$, $(p, m)=1$, and π_p, π_m be subgroups of π of orders p^d, m respectively. Let k be a field of characteristic p, and containing all the m-th roots of unity. Then, if $f_1,\cdots,f_m$ denote all the primitive idempotents in $k\pi_m$, all $k\pi f_j$'s are indecomposable $k\pi$-modules and they have the same rank over k.

Proof. $k\pi f_j\cong k\pi_p\otimes_k k\pi_m f_j\cong k\pi_p$ as a $k\pi_p$-module, so it is indecomposable because $k\pi_p$ is a local ring. It is also trivial about the ranks.

Theorem 1. Let R be a commutative ring with identity such that any zero divisor is in the Jacobson radical, and also we assume that the total quotient ring K of R is semi-local. Let π be a finite group of order n. We assume that any prime number dividing n is nonunit in R. Then for any finitely generated projective $R\pi$-module P, $K\otimes_R P$ is a free $K\pi$-module.

Proof. Let p be a prime dividing n, and $\mathfrak{p}$ a maximal ideal of R which contains p. $R_\mathfrak{p}$ denotes the localization of R at $\mathfrak{p}$. Then $R_\mathfrak{p}\otimes_R P$ is $R_\mathfrak{p}\pi$- therefore $R_\mathfrak{p}\pi_p$-projective for a p-Sylow subgroup π_p of π. But $R_\mathfrak{p}\pi_p$ is a local ring because $\mathfrak{p}R_\mathfrak{p}\pi_p$ is contained in the Jacobson radical, and $(R/\mathfrak{p})\pi_p$ is local. So $R_\mathfrak{p}\otimes_R P$ is $R_\mathfrak{p}\pi_p$-free, and especially $K\otimes_R P=K\otimes_{R_\mathfrak{p}}R_\mathfrak{p}\otimes_R P$ has a rank divisible by the order of π_p. (The assumption on R guarantees that K is also the total quotient ring of $R_\mathfrak{p}$.) Then the rank of $K\otimes_R P$ is divisible by n. By Lemma 4 we see that we need only to prove the theorem in the cyclic case. Now we assume that π is a cyclic group of order $n=p^d m$, $(p, m)=1$, and π_p, π_m denote the subgroups of orders p^d, m respectively. We proceed by the induction on the order of π. If $\pi=(1)$, $K\otimes_R P=K\otimes_{R_\mathfrak{p}}(R_\mathfrak{p}\otimes_R P)$ is K-free ($\mathfrak{p}$ is any maximal ideal of

R). We assume that $K\otimes_R P$ is $K\pi_m$-free. Let $\mathfrak{p}$ be a maximal ideal containing p. Then by Lemma 3 $R_{\mathfrak{p}}\otimes_R P$ is $R_{\mathfrak{p}}\pi_m$-free. By [2, Lemma 2.4], $R_{\mathfrak{p}}\otimes_R P$ is $R_{\mathfrak{p}}\pi$-free if and only if $(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})\otimes_R P=(R/\mathfrak{p})\otimes_R P$ is $(R/\mathfrak{p})\pi$-free. We put $k=R/\mathfrak{p}$. By Noether-Deuring theorem we may assume k contains all the m-th roots of unity. Then by Lemma 5, $k\otimes_R P=\sum_j a_j\cdot k\pi f_j$ is a decomposition to the indecomposable components, where a_j denotes the number of components isomorphic to $k\pi f_j$. By considering the ranks over k, all a_j's are equal as $k\otimes_R P$ is $k\pi_m$-free. Therefore we have $k\otimes_R P=a\cdot k\pi$, and this completes the proof.

Examples. R satisfies the assumptions of the theorem in the following cases.

1) $R=A[X_1, X_2, \cdots]$ is a polynomial ring over the ring A of algebraic integers.

2) R is a ring such that no prime $p\mid n$ is a unit, and any zero divisor is nilpotent. Then K is a local ring.

Next we assume that R is a noetherian ring. Then its total quotient ring is semi-local by [4, IV, Corollary 3 of Theorem 11]. So we need only the trivial conditions on R by taking Serre's theorems [5] into account, i.e. we have Theorem 2 below.

Lemma 6. Let R be a commutative indecomposable noetherian ring. Let π be a finite group of order n, and any prime factor of n be non-unit in R. Then $R\pi$ is also indecomposable.

Proof. We assume that $R\pi=A\oplus B$ is a decomposition to the left ideals. Let p be a prime factor of n, π_p be a p-Sylow subgroup of π, and $\mathfrak{p}$ be a prime ideal containing p. Then $R_{\mathfrak{p}}\pi\cong A_{\mathfrak{p}}\oplus B_{\mathfrak{p}}$, and $A_{\mathfrak{p}}$ is $R_{\mathfrak{p}}\pi_p$-projective, so $R_{\mathfrak{p}}\pi_p$-free. Therefore its rank over $R_{\mathfrak{p}}$ is a multiple of the order of π_p. As we assumed that R is indecomposable, its rank depends neither on $\mathfrak{p}$, nor on p [5, Proposition 4]. Hence it is divisible by the order of π, and A must be equal to $R\pi$.

Lemma 7. Let R be a commutative indecomposable noetherian ring, and K be its total quotient ring. Then for any projective R-module P, $K\otimes_R P$ is K-free.

Proof. Put $K=K_1\oplus\cdots\oplus K_r$, where K_i is indecomposable ideal of K. Then $K_i\otimes_R P$ is K_i-free by [5, Proposition 6]. So we need only to prove that the rank of $K_i\otimes_R P$ over K_i does not depend on i. We take a prime ideal $\mathfrak{m}_i$ of K_i for each i. Then $\mathfrak{p}_i=(K_1\oplus\cdots\oplus K_{i-1}\oplus \mathfrak{m}_i\oplus K_{i+1}\oplus\cdots\oplus K_r)\cap R$ is a prime ideal of R, and there exists a monomorphism of $R_i=R_{\mathfrak{p}_i}$ into $K_{i,\mathfrak{m}_i}$ by $x\to xe_i$, where e_i is the identity element of K_i. As R_i is a local ring, $R_i\otimes_R P$ is R_i-free. So its rank is equal to that of $K_{i,\mathfrak{m}_i}\otimes_R P$, and then equal to that of $K_i\otimes_R P$. As R is indecomposable, the rank of $R_i\otimes_R P$ is independent of i, so is the rank of $K_i \otimes_R P$.

Lemma 8. Let R be a commutative indecomposable noetherian ring. Let π be a finite abelian group of order n, and any prime factor of n be non-unit in R. Let K denote the total quotient ring of R. Then for any projective $R\pi$-module P, $K \otimes_R P$ is $K\pi$-free.

Proof. Let Q be the total quotient ring of $R\pi$. Then $R\pi \subset K\pi \subset Q$ holds, and $Q \otimes_{R\pi} P$ is Q-free by Lemma 7. By considering the isomorphism $K \otimes_R P \cong K\pi \otimes_{R\pi} P$, it suffices to show the latter is $K\pi$-free. Put $K\pi = K_1 \oplus \cdots \oplus K_r$, where K_i is indecomposable ideals. Then $Q \cong Q \otimes_{K\pi} K\pi \cong \sum_i Q \otimes_{K\pi} K_i$ holds, and no $Q \otimes_{K\pi} K_i$ is zero because $K\pi \subset Q$. In the direct decomposition $K\pi \otimes_{R\pi} P \cong K_1 \otimes_{R\pi} P \oplus \cdots \oplus K_r \otimes_{R\pi} P$, each $K_i \otimes_{R\pi} P$ is K_i-projective, so it is K_i-free because K_i is indecomposable semi-local ring. The rank of which over K_i is equal to that of $Q \otimes_{K\pi} K_i \otimes_{R\pi} P$ over $Q \otimes_{K\pi} K_i$. But the latter is independent of i, because $Q \otimes_{R\pi} P \cong \sum_i Q \otimes_{K\pi} K_i \otimes_{R\pi} P$ is Q-free. Hence $K \otimes_R P \cong K\pi \otimes_{R\pi} P$ is $K\pi$-free.

Theorem 2. Let R be a commutative ring with identity which is noetherian and indecomposable. Let K denote its total quotient ring. Let π be a finite group of order n, and we assume that any prime factor of n is non-unit in R. Then for any finitely generated projective $R\pi$-module P, $K \otimes_R P$ is a free $K\pi$-module.

Proof. By Lemma 8, $K \otimes_R P$ is $K\pi'$-free for any cyclic subgroup π' of π. So we need only to prove the last assumption of Lemma 4. Let p be a prime factor of n, and $\mathfrak{p}$ be a prime ideal containing p. Then $R_{\mathfrak{p}} \otimes_R P$ is $R_{\mathfrak{p}}\pi_p$-free as in the proof of Theorem 1, where π_p is a p-Sylow subgroup of π. Let $K_{\mathfrak{p}}$ denote the total quotient ring of $R_{\mathfrak{p}}$. Then $K_{\mathfrak{p}} \otimes_R P$ is $K_{\mathfrak{p}}\pi_p$-free. There is a natural homomorphism of K into $K_{\mathfrak{p}}$. So the rank of $K \otimes_R P$ over K is equal to that of $K_{\mathfrak{p}} \otimes_R P = K_{\mathfrak{p}} \otimes_K K \otimes_R P$ over $K_{\mathfrak{p}}$. The rank of $K_{\mathfrak{p}} \otimes_R P$ is divisible by the order of π_p, so is the rank of $K \otimes_R P$. As p is any prime factor of n, the rank of $K \otimes_R P$ is divisible by the order of π. Therefore we complete the proof of the theorem by Lemma 4.

References

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