3. A Remark on Components of Ideals in Noncommutative Rings

By Hidetoshi MARUBAYASHI
Department of Mathematics, Yamaguchi University

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Let R be a noncommutative ring, let A be an ideal[1] in R, and let M be a non-empty m-system in the sense of McCoy.[2] The right upper and the right lower isolated M-components of A, in the sense of Murdoch,[3] will be denoted by $U(A, M)$ and $L(A, M)$ respectively. In [3], D. C. Murdoch has obtained the following result:

If the ascending chain condition holds in the residue class ring R/A, then $L^n(A, M)$[4] $= U(A, M)$ for some positive integer n.

The aim of this short note is to prove that $n=1$ under an assumption which is weaker than that of Murdoch.

Theorem. *Let $S[A, M]$ be the set of right ideal quotients AB^{-1},[5] where B runs over all ideals meeting the m-system M. Suppose that R satisfies the ascending chain condition for elements of $S[A, M]$. Then $L(A, M)=U(A, M)$.*

Proof. This result will follow from Theorem 5 of [3], if it can be shown that $L(A, M)=L(L(A, M), M)$. By the assumption, there exists a maximal element A_0 in $S[A, M]$ such that $A_0=AB_0^{-1}$ for an ideal B_0 which meets M.

(i) We shall prove that $A_0=L(A_0, M)$. Let x be any element of $L(A_0, M)$. Then we have $xRm\subseteqq A_0$ for some $m\in M$. Hence $x\in A_0(m)^{-1}=A((m)RB_0)^{-1}$, where (m) is the principal ideal generated by m. Now we shall show that $(m)RB_0$ meets the m-system M. For, if $(m)RB_0$ does not meet M, then there exists a prime ideal P, by Lemma 4 of [2], such that $P\supseteqq(m)RB_0$ and $P\cap M=\phi$. Hence we have $m\in P$ or $B_0\subseteqq P$. This is a contradiction. Therefore the maximal property of A_0 implies that $A_0=A_0(m)^{-1}$. Thus $A_0\supseteqq L(A_0, M)$. The converse inclusion is obvious. Hence we have $A_0=L(A_0, M)$.

(ii) We shall prove that $A_0=L(A, M)$. By the definition, we have $A_0\subseteqq L(A, M)$. Let x be any element in $L(A, M)$. Then we have $xRm\subseteqq A$ for some $m\in M$. Thus $xRm\subseteqq A_0$. Hence we obtain $x\in A_0(m)^{-1}$. By the above discussion, it is clear that $A_0=A_0(m)^{-1}$. We have therefore $A_0=L(A, M)$. This completes the proof.

1) The term *"ideal"* will mean *"two-sided ideal"*.
2) Cf. [2].
3), 4), 5) Cf. [3].

Remark. $S[A, M]$ has a unique maximal ideal. Because, let AB^{-1} and AC^{-1} be any two maximal ideals of $S[A, M]$, where B and C are ideals which meet the m-system M. Then the product BC also meets M. Therefore $A(BC)^{-1}$ is a member of $S[A, M]$. It is clear that $A(BC)^{-1}$ contains AB^{-1} and AC^{-1}. Hence by the maximality of AB^{-1} and AC^{-1}, we obtain that $AB^{-1}=A(BC)^{-1}=AC^{-1}$.

Corollary. Suppose that R satisfies the ascending chain condition for ideals in R. Let P be a minimal prime divisor of an ideal A. Then $L(A, P)$[6] is a right primal ideal with adjoint P in the sense of Barnes.[7]

Proof. Since $L(A, P)=U(A, P)$, each element of $C(P)$ is right prime[8] to $L(A, P)$ by Lemma 3 of [3]. Hence this result will follow if it can be shown that P is not right prime[9] to $L(A, P)$. P is obviously a minimal prime divisor of $L(A, P)$. If P is right prime to $L(A, P)$, then, by Corollary to Theorem 10 in [3], there exists a minimal prime divisor $P'(\neq P)$ of $L(A, P)$ which is not right prime to $L(A, P)$. Hence there exists an element b in $C(P)$ which is not right prime to $L(A, P)$. This is a contradiction.

References

[1] W. E. Barnes: Primal ideals and isolated components in noncommutative rings. Trans. Amer. Math. Soc., **82**, 1-16 (1956).

[2] N. H. McCoy: Prime ideals in general rings. Amer. J. Math., **71**, 823-833 (1948).

[3] D. C. Murdoch: Contributions to noncommutative ideal theory. Canadian J. Math., **4**, 43-57 (1952).

6) $L(A, P)$ and $U(A, P)$ will denote $L(A, C(P))$ and $U(A, C(P))$ respectively, where $C(P)$ is the complement of P in R.

7) Cf. [1].

8), 9) Cf. [3].