1. On the Decomposition of Regular Representation of the Lorentz Group on a Hyperboloid of one Sheet

By Takuro SHINTANI

Department of Mathematics, University of Tokyo

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1. Let G_n be the subgroup of $GL(n+1, R)$ consisting of elements which leave invariant the quadratic form $-x_0^2+x_1^2+\cdots+x_n^2$, and G_n^+ be the connected component of G_n. Let X be the hyperboloid of one sheet in R^{n+1} with the equation $x_0^2-x_1^2-\cdots-x_n^2=-1$. G_n naturally operates on X and the measure on X defined by $dx=\frac{dx_1\cdots dx_n}{|x_0|}$ is invariant under the action of G_n. Let $L^2(X)$ be the Hilbert space of functions on X which are square integrable with respect to this measure. Then we get the unitary representation π of G_n^+ on $L^2(X)$ defined as follows: $(\pi(g)f)(x)=f(xg)$, $g\in G_n^+$, $f\in L^2(X)$, $x\in X$. We denote the corresponding representation of the universal enveloping algebra of Lie algebra of G_n^+ also by π. In this note we decompose π into direct sum of irreducible representations. In the following, we use the notations defined in R. Takahashi [1] Chap. I, §1, §2 without further reference.

2. For any complex number s we define the representations $(U^s, \mathcal{H})$ of G_n^+ as follows:

Let H be the linear space of C^∞ functions on K which are invariant under left translations of M and $(U^s(g)f)(k)=e^{-st(k,g)}f(kg)$, $f\in\mathcal{H}$, where kg and $t(k, g)$ is defined uniquely by the relations $kg=a_{t(k,g)}n\,kg$, $a_{t(k,g)}\in A$, $n\in N$, and $kg\in K$. In the following for special value of s we define the positive (in general not definite) inner product $(\ ,\ )_s$ in $\mathcal{H}$ so that U_s becomes unitary, and we get unitary representation $(U_s, \mathcal{H}_s)$ where $\mathcal{H}_s$ is the completion of $\mathcal{H}$ with respect to the norm $\|\ \|_s$ defined by inner product $(\ ,\ )_s$.

When $s=-\frac{n-1}{2}+i\rho$, $\rho\in R$, we define for any $\varphi, \psi\in\mathcal{H}$, $(\varphi, \psi)_s=\int\varphi\bar{\psi}\,dk$ where dk is the normalized Haar measure of K. For any φ, $\psi\in\mathcal{H}$, and s, $\left(Re\, s<-\frac{n-1}{2}\right)$ we put

$$I_s(\varphi, \psi)=C_s\int\langle vk_1, vk_2\rangle^{-(n-1+s)}\varphi(k_1)\overline{\psi(k_2)}\,dk_1\,dk_2$$

where $C_s=\frac{\sqrt{\pi}\,\Gamma(-s)}{2^{-(1+s)}\Gamma\left(\frac{n}{2}\right)\Gamma\left(-\left(s+\frac{n-1}{2}\right)\right)}$ and $v=(1, -1, 0, \cdots, 0)$.

The inner product $\langle x, y\rangle$ is defined as $\langle x, y\rangle = x_0y_0 - x_1y_1 - \cdots - x_ny_n$ for any two vectors $x=(x_0, x_1, \cdots, x_n)$ and $y=(y_0, y_1, \cdots, y_n)$. By the analytic continuation regarded as the function of s, $I_s(\varphi, \psi)$ becomes a meromorphic function on the whole s-plane and has simple poles at $s=0, 1, 2, \cdots$ and it is proved that $I_s(\varphi, \varphi)\geqslant 0$ for $-(n-1)<s<0$ and $\operatorname*{res.}_{s=l} (-1)^{l-1}I_s(\varphi, \varphi)\geqslant 0$ for $l=0, 1, 2, \cdots$. We put $(\varphi, \psi)_s = I_s(\varphi, \psi)$ when $-(n-1)<s<0$ and $(\varphi, \psi)_s = \operatorname*{res.}_{s=l} (-1)^{l-1}I_s(\varphi, \psi)$ when $s=l=0, 1, 2, \cdots$. Then it is proved that $(U_s, \mathcal{H}_s)$ $\left(\operatorname{Re} s = -\dfrac{n-1}{2}\right.$ or $-(n-1)<s<0$ or $\left. s=0, 1, 2, \cdots\right)$ are unitary representations and are irreducible when $n\neq 2$. When $n=2$, every $(U_l, \mathcal{H}_l)$ for $l=0, 1, 2, \cdots$ is decomposed into the direct sum of two irreducible representations and other (U_s, H_s) are irreducible. (For details see Takahashi [1] and Gelfand-Graev-Vielenkin [1].)

3. We denote by $C_c^\infty(X)$ the space of indefinitely differentiable functions on X with compact support. For any $f\in C_c^\infty(X)$ and $s\in C$, $(\operatorname{Re} s > -1)$ we define $\mathcal{F}_s^1(f)$, $\mathcal{F}_s^2(f)\in\mathcal{H}$ as follows:

$$\mathcal{F}_s^1(f)(k) = \int_{\langle v, x\rangle>0} f(xk)\langle v, x\rangle^s dx,$$

$$\mathcal{F}_s^2(f)(k) = \int_{\langle v, x\rangle<0} f(xk)\{-\langle v, x\rangle\}^s dx,$$

where $v=(1, -1, 0, \cdots, 0)$. Then we get the following.

Lemma. a) For any fixed $k\in K$, $\mathcal{F}_s^i(k)$ $(i=1, 2)$ is continued to a meromorphic function on the whole s-plane which has simple poles at $s=-1, -2, \cdots$.

b) For a fixed s $(s\neq -1, -2, \cdots)$

$$\mathcal{F}_s^i(\pi(g)f) = U_s(g)F_s^i(f),\ i=1, 2,\ g\in G_n^+.$$

c) Let $\operatorname*{res.}_{s=-j} \mathcal{F}_s(f)$ be the element of $\mathcal{H}$ defined by $(\operatorname*{res}_{s=-j} \mathcal{F}_s(f))(k) = \operatorname*{res.}_{s=-j} (\mathcal{F}_s(f)(k))$, then

$$\operatorname*{res.}_{s=-j} \mathcal{F}_s^i(\pi(g)f) = U_{-j}(g)(\operatorname*{res.}_{s=-j} \mathcal{F}_s^2(f))$$
$$i=1, 2;\ k=1, 2, \cdots, g\in G_n^+.$$

We define for any $f\in C_c^\infty(X)$ and s, $\left(\operatorname{Re} s = -\dfrac{n-1}{2}\right.$ or $s=-j, \left[\dfrac{n}{2}-1\right] \geqslant j \geqslant +1$ or $\left. s=l=0, 1, 2, \cdots\right)$, $\Pi_s^i(f)\in\mathcal{H}$ $(i=1, 2)$ as $\Pi^i_{-(n-1)/2+i\rho}(f)$, $=\mathcal{F}^i_{-(n-1)/2+i\varphi}(f)$, $\rho\in R$, $i=1, 2$, $\Pi^1_{-j}(f) = \Pi^2_{-j}(f) = \operatorname*{res.}_{s=-j} F_s^1(f)$, $j=1, \cdots, \left[\dfrac{n}{2}-1\right]$, $\Pi_l^1(f) = \Pi_l^2(f) = \mathcal{F}_e^1(f)$.

Then we have the following.

Theorem 1. For any $f\in C_c^\infty(X)$ we have the following equality

$$\int |f(x)|^2 dx = \int_0^\infty \omega_n(\rho)\{\|\Pi^1_{-(n-1)/2+i\rho}(f)\|^2_{-(n-1)/2+i\rho} + \|\Pi^2_{-(n-1)/2+i\rho}(f)\|^2_{-(n-1)/2+i\rho}\}\, d\rho$$

$$+\sum_{1\leqslant j\leqslant[n/2-1]}\tilde{\omega}_n(-j)\,||\,\Pi'_{-j}(f)\,||^2_{-j}$$
$$+\sum_{l\geqslant 0}\tilde{\omega}_n(l)\,||\,\Pi'_l(f)\,||^2_l,$$

where ω_n and $\tilde{\omega}_n$ are defined as follows:
When $n=2m$,

$$\omega_n(\rho)=(2^{n-1}\pi^m(m-1)!)^{-1}\rho\,\mathrm{th}\,\pi\rho\Big(\rho^2+\frac{1}{4}\Big)\Big(\rho^2+\frac{9}{4}\Big)\cdots\Big(\rho^2+\Big(m-\frac{3}{2}\Big)^2\Big),$$
$$\tilde{\omega}_n(-j)=(2^{n-1}\pi^{m-1}(m-1)!)^{-1}(n-2j-1)(j-1)!(n-2-j)!(j=1,\cdots,m-1)$$
$$\tilde{\omega}_n(l)=(2^{n-1}\pi^{m-1}(m-1)!)^{-1}(2l+n-1)\frac{(l+n-2)!}{l!}$$

when $n=2m+1$,

$$\omega_n(\rho)=(2^m\pi^{m+1}(2m-1)!!)^{-1}\rho^2(1+\rho^2)\cdots\{(m-1)^2+\rho^2\},$$
$$\tilde{\omega}_n(-j)=(2^m\pi^m(2m-1)!!)^{-1}(n-2j-1)(j-1)!(n-2-j)!\ j=1,\cdots,m-1,$$
$$\tilde{\omega}_n(l)=(2^m\pi^m(2m-1)!!)^{-1}(2l+n-1)\frac{(l+n-2)!}{l!}.$$

4. Let $\Delta=-\pi(\Omega)$, where Ω is defined in R. Takahashi [1] p. 327. Δ is the Casimir operators on X which is known to be essentially self-adjoint on $L^2(X)$. Let $P_0=(0,1,\cdots,0)\in X$, G_{n-1} be the subgroup of G_n consisting of elements which leave invariant the point p_0, and let $X'=\{x\in X,\ x_2\neq\pm 1\}$.

Our proof of theorem 1 depends on the following two theorems.

Theorem 2. A. For any $\lambda\in C(\mathrm{Im}\,\lambda\neq 0)$ there is a unique distribution T_λ on X satisfying the following three conditions.

1. T_λ is invariant under G_{n-1} that is, $T_\lambda(\pi(g)f)=T_\lambda(f)$ for any $g\in G_{n-1}$, $f\in C_c^\infty(X)$.
2. $(\Delta-\lambda)T_\lambda=\delta(p_0)$ ($\delta(p_0)$ is the Dirac measure concentrated to p_0)
3. When we put $s=\dfrac{n-1+\sqrt{(n-1)^2+4\lambda}}{2}$ (square root is taken so that its real part is positive) $\limsup_{|x_2|\to\infty}|\,|x_2|^s T_\lambda|<\infty$ (It is proved that, T_λ, satisfying conditions 1, 2, coincides in X' with analytic function depending only on x_2.)

B. $(\Delta-\lambda)^{-1}$, resolvent of Δ, is expressed by T_λ as follows:

$$((\Delta-\lambda)^{-1}f)(x)=T_\lambda(\pi_{(g_x)}f),\ x\in X,\ f\in C_C^\infty(X).$$

Here g_x is any element in G_n such that $p_0g_x=x$.

Let $F(x)$ be a function defined on X^- such that for any $\varphi\in C_C^\infty(X)$ the integral $F(\varphi,s)=\int F(x)\varphi(x)\,|\,x_2^2-1\,|^s\,dx$ converges absolutely for s whose real part is sufficiently large, and $F(\varphi,s)$, as a function of s can be continued analytically to the neighbourhood of $s=0$ and at $s=0$ has Laurent expansion

$$F(\varphi,s)=\cdots+\frac{\mu_{-1}}{s}(F,\varphi)+\mu_0(F,\varphi)+\cdots.$$

Then we define

$$(\text{Fin. } F)(\varphi)=\mu_0(F, \varphi) \quad \text{and}$$
$$(\text{Res. } F)(\varphi)=\mu_{-1}(F, \varphi).$$

Theorem 3. The distribution T_λ, defined in Theorem 2 has the following expression:

Case I. $n=2m+1$.

$$T_\lambda(f)=(\text{fin. } {}_nF_\lambda)(f)$$

where ${}_nF_\lambda$ is a function defined as follows:

$${}_nF_\lambda(x)=0 \qquad \text{when } x_2<-1$$

$${}_nF_\lambda(x)=(-2^{m+2}(s-m)\pi^m\, i\sin\pi(s-m))^{-1}$$
$$\times\frac{d^m}{dx_2^m}\{(-x_2+i\sqrt{1-x_2^2})^{s-m}-(-x_2+i\sqrt{1-x_2^2})^{-(s-m)}\} \text{ when } |x_2|<1$$

(s is defined in Th. 2, $0<\arg(-x_2+i\sqrt{1-x_2^2})<\pi$)

$${}_nF_\lambda(x)=(-2^{m+1}(s-m)\pi^m)^{-1}\frac{d^m}{dx_2^m}(x_2+\sqrt{x_2^2-1})^{-(s-m)} \qquad \text{when } x_2>1$$

Case II. $n=2$.

$$T_\lambda(f)=(\text{fin. } {}_n\tilde{F}_\lambda)(f)=\int f(x)\tilde{F}_\lambda(x)\,dx$$

Case III. $n=2m$, $m>1$.

$$T_\lambda(f)=(\text{fin. } {}_n\tilde{F}_\lambda)(f)$$
$$+(-1)^m\left\{\left(\frac{n}{2}-2\right)!\,2^{n-2}\pi^{m-1}\right\}^{-1}\prod_{k=1}^{m-1}\{2k(2k+1-n)-\lambda\}$$
$$\times\sum_{k=0}^{m-2}4^k k!\frac{\Gamma(m-1)}{\Gamma(m-1-k)}\left\{\prod_{l=1}^{k+1}(2l(2l+1-n)-\lambda)^{-1}\right\}$$
$$\times(\text{res. } (|x_2^2-1|^{-(k+1)}\chi)f,$$

where χ is a characteristic functions of subset R of X defined as follows:

$$R=\{x\in x,\ x_2>1\}$$

And ${}_n\tilde{F}_\lambda$ is defined as follows.

$${}_n\tilde{F}_\lambda(x)=(-(2\pi)^m\sin\pi s)^{-1}\left(\frac{d^{m-1}}{dx_2^{m-1}}Q_{s-m}\right)(-x_2) \qquad \text{when } x_2<-1$$

$${}_n\tilde{F}_\lambda(x)=(-2(2\pi)^m\sin\pi s)^{-1}\left\{\left(\frac{d^{m-1}}{dx_2^{m-1}}Q_{s-m}\right)(-x_n+i0)\right.$$
$$\left.+\left(\frac{d^{m-1}}{dx_2^{m-1}}Q_{s-m}\right)(-x_n-i0)\right\} \qquad \text{when } |x_2|<1$$

$${}_n\tilde{F}_\lambda(x)=(-(2\pi)^m)^{-1}\cot\pi s\left(\frac{d^{m-1}}{dx^{m-1}}Q_{s-m}\right)(x_2) \qquad \text{when } x_2>1.$$

(Q_ν is a Legendre function of second kind.)

When we substitute the expression of $(\Delta-\lambda)^{-1}$ obtained by Theorems 2 and 3 to the well-known formula

$$\int |f(x)|^2 dx$$
$$= \lim_{\substack{\alpha \to +\infty \\ \beta \to -\infty}} \int_\beta^\alpha \left[\frac{1}{2\pi i} \lim_{v \downarrow 0} \int \overline{f(x)} \{(\Delta - (u+iv))^{-1} f(x) - (\Delta - (u-iv))^{-1} f(x)\} dx \right] du,$$

we get Theorem 1.

The detailed proof of this note will appear elsewhere. When $n=3$, our problem is solved with different method by Gelfand-Graev [1] in more refined form (see also Gelfand-Graev-Vielenkin [1]).

References

Gelfand-Graev [1]: Application of method of horispheres to spectral analysis of functions on real and imaginary Lobatschevskij spaces. Trudy, **11**, 243-308 (1962).

Galfand-Graev-Vielenkin [1]: Generalized functions. V.

R. Takahashi: Sur les représentations unitaires des groupes de Lorentz généralisées. Bull. Soc. Math. France, **91**, 289-433 (1963).

After the preparation of this paper, the author has found that V. F. Molchanov has obtained the almost same result of the same problem through different method. "Harmonic analysis on a hyperboloid of one sheet." Doklady Akad. Nauk, S.S.S.R., **171** (4), 794-797 (1966). In that Dr. Molchanov has missed the contribution of the complementary series $\pi'_{-j}(f)\left(j=1, \cdots, \left[\frac{n}{2}-1\right]\right)$ in our terminology.