31. A Note on Regularity of Null Solutions

By Kenzo SHINKAI

Department of Mathematical Sciences, University of Osaka Prefecture

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The object of this note is to show that every null solution of partial differential operators of a certain class belongs to the Gevrey class $G_x(0)$ with respect to the space variable. This gives a partial answer for Kumano-go's problem: "Is it possible to construct a null solution such that its derivative of some order has the discontinuity with respect to space-variables at some point (t_0, x_0)?" H. Kumano-go [1].

Our results are stated in the following

Theorem. *Let $L(\lambda, \zeta)$ be a polynomial in λ and ζ with constant coefficients and have the form*

(1) $$L(\lambda, \zeta)=\sum_{\substack{0\leq j\leq j_0\\ 0\leq k\leq k_0}} a_{j,k}\lambda^j\zeta^k, \qquad j_0>0,\ k_0>0,\ \text{and}\ a_{j_0,k_0}=1.$$

Let u be a distribution solution of the equation

(2) $$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)u=0$$

in R^2. If u vanish when $t\leq 0$ and if there exist an open neighbourhood of the t-axis where $(\partial^k/\partial x^k)u$ for $0\leq k\leq k_0-1$ be functions which are continuously differentiable with respect to t for j_0 times.

Then u is a continuous function of t and x which is entire with respect to x, and satisfies the following inequality

(3) $$\left|\frac{\partial^k}{\partial x^k}u(t, x)\right|\leqq C_0^{k+1}e^{c_1|x|}, \quad k=0, 1, 2, \cdots, \quad t\leqq T,$$

where C_0 and C_1 are constants which are independent of k and x.

Remark 1. The telegraph equation

$$u_{tt}=u_{xx}-r^2u \quad (r=\text{constant})$$

can easily be transformed into an operator of the form (1) by introducing new independent variables

$$\xi=t+x,\ \eta=t-x.$$

Remark 2. Let $P(\lambda, \zeta)$ be a polynomial in λ and ζ, and its degree with respect to ζ be equal to $K>0$. We can then write

(4) $$P(\lambda, \zeta)=Q_0(\lambda)\prod_{k=1}^{K}(\zeta-\zeta_k(\lambda)),$$

where every ζ_k for some positive integer p_k is an analytic function of λ^{-1/p_k} when $|\lambda|>C$, with no essential singularity at infinity, that is,

(5) $$\zeta_k(\lambda)=\sum_{n=N_k}^{\infty}a_n(\lambda^{-1/p_k})^n.$$

We assume that

(6) $$N_1/p_1 \leqq N_2/p_2 \leqq \cdots \leqq N_K/p_K.$$

Then the following properties (i)~(iii) are equivalent:

(i) $N_K \leqq 0$.

(ii) $|\zeta_k(\lambda)|$ is bounded for every k when $|\lambda| \to \infty$.

(iii) P has the same form as that of L defined in the Theorem.

that (Proof of Remark 2.) It is obvious that (i) and (ii) are equivalent. In (6) we may assume that there is an integer h such that $1 \leqq h \leqq K$ and

(7) $$N_1/p_1 \leqq \cdots \leqq N_{h-1}/p_{h-1} < N_h/p_h = N_{h+1}/p_{h+1} = \cdots = N_K/p_K.$$

If we write

$$P(\lambda, \zeta) = Q_0(\lambda)\zeta^k + Q_1(\lambda)\zeta^{k-1} + \cdots + Q_K(\zeta)$$

and set $\deg Q_k(\lambda) =$ degree of $Q_k(\lambda)$, then we have

(8) $$\deg Q_{k-h+1}(\lambda) = \deg Q_0(\lambda) + (K-h+1)N_K/p_K.$$

Now if P satisfies (iii), we have

$$\deg Q_0(\lambda) \geqq \deg Q_{K-h+1}(\lambda),$$

and with (8) we have $N_K \leqq 0$.

Conversely, if P does not satisfy (iii), that is,

$$\deg Q_0(\lambda) < \deg Q_k(\lambda) \qquad \text{for some } k,$$

then (7) gives

$$0 < \deg Q_k(\lambda) - \deg Q_0(\lambda) \leqq N_{K-k+1}/p_{K-k+1} + \cdots + N_K/p_K$$
$$\leqq k(N_K/p_K).$$

Hence we have $N_K > 0$.

Remark 3. We use the same notation as in Remark 2. In the note [2] H. Kumano-go and the author proved that when $N_1/p_1 \leqq 0$, the null solution of the equation $Pu=0$ are not able to belong to $G_x(N_1/p_1 - \varepsilon)$ for any $\varepsilon > 0$, and there exists a null solution which belongs to $G_x(N_1/p_1)$.

The above theorem and Remark 2. mean that every null solution belongs to $G_x(0)$ when $N_K/p_K \leqq 0$. But the converse is not ture. It is a problem still pending to deside the operators whoes null solutions are all in $G_x(0)$.

Lemma 1. *Let L and u satisfy the condition in the theorem and let*

(9) $$\frac{\partial^k}{\partial x^k} u(t, 0) = 0, \quad 0 \leqq k \leqq k_0 - 1.$$

Then u vanishes identically.

Proof. The equation (2) gives

(10) $$Q_0\left(\frac{\partial}{\partial t}\right) \frac{\partial^{k_0}}{\partial x^{k_0}} u(t, 0) = 0,$$

where $Q_0(\lambda)$ is the polynomial defined by

$$L(\lambda, \zeta) = Q_0(\lambda)\zeta^{k_0} + Q_1(\lambda)\zeta^{k_0-1} + \cdots + Q_{k_0}(\lambda).$$

Since (10) is an ordinary differential equation with respect to $(\partial^k/\partial x^{k_0})u(t, 0)$ which vanishes when $t \leqq 0$, we have

$$\frac{\partial^{k_0}}{\partial x^{k_0}}u(t,0)=0.$$

By differentiation with respect to x of the equation (2), we have

$$\frac{\partial^k}{\partial x^k}u(t,0)=0 \qquad \text{for all } k\geqq 0.$$

If we set

$$\tilde{u}=\begin{cases} u & \text{when } t>0,\ x>0,\\ 0 & \text{otherwise,}\end{cases}$$

then $\tilde{u}$ is a solution of the equation (2) in R^2. Now we can apply the uniqueness theorem of Holmgren and we get

$$\tilde{u}=0.$$

This proves the lemma.

Let H be the integral operator defined by the following identity:

(11) $$H[u(t,x)]=L\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right)\int_0^t d\tau\int_0^x \frac{(t-\tau)^{j_0-1}(x-y)^{k_0-1}}{(j_0-1)!\,(k_0-1)!}u(\tau,y)dy-u(t,x).$$

Then we have the following

Lemma 2. *Let $f(t,x)$ be a polynomial in x whoes coefficients are continuous functions of t. Then the series*

(12) $$\sum_{n=0}^{\infty}H^n[f(t,x)]$$

converges absolutely to a continuous function $w(t,x)$ which is entire with respect to x and satisfies the inequality same to (3).

Proof. In order to simplify the statements we assume that $f(t,x)=f(t)x^p$, $f(t)$ is a continuous function. As $H^l[f(t,x)]$ is a polynomial in x, setting $g_{k,l}(t)$ as the coefficient of x^k in $H^l[f(t,x)]$, we have

(14) $$|g_{k,l}(t)|\leqq (j_0k_0)^l a^l T^{j'}p_0 p!/(j'!\,k!), \text{ when } 0\leqq t\leqq T,$$

where

$$p_0=\max\{|f(t)|;\ 0\leqq t\leqq T\},$$
$$a=\max\{|a_{j,k}|;\ a_{j,k} \text{ are coefficients of } L\},$$
$$j'=\max\{0,\ l-k+p\}.$$

We can choose a constant $C\geqq 1$ which satisfies the following inequality:

$$(j_0k_0)^l a^l T^{j'}p_0 p!\leqq T^{-k}C^{l+1}.$$

Then we have

$$|g_{k,l}(t)|\leqq \frac{T^{-k}C^{l+1}}{k!\,j'!}.$$

If $k\geqq p$, we have

$$\begin{aligned}\left|\sum_{l=0}^{\infty}g_{k,l}(t)\right| &\leqq \sum_{l=0}^{\infty}|g_{k,l}(t)|\\ &\leqq \frac{T^{-k}}{k!}\sum_{l=0}^{\infty}\frac{C^{l+1}}{j'!}\\ &=\frac{T^{-k}}{k!}\left\{\sum_{l=0}^{k-p-1}C^{l+1}+C^{k-p}\sum_{l=k-p}^{\infty}\frac{C^{l-k+1}}{(l-k+1)!}\right\}\\ &\leqq \frac{T^{-k}}{k!}\{(k-p)+e^C\}C^{k-p}.\end{aligned}$$

Hence we have a constant C_0 such that

$$\left|\sum_{l=0}^{\infty} g_{k,l}(t)\right| \leqq C_0^{k+1}/k! \qquad \text{for every } k \geqq 0. \tag{15}$$

This means that the series (12) is an absolutely convergent power series of x and the absolute value of the coefficient of x^k is dominated by $C_0^{k+1}/k!$, which proves the lemma.

Proof of the theorem. If we take the function

$$f_0(t, x) = \sum_{k=0}^{k_0-1} \frac{x^k}{k!} \frac{\partial^k}{\partial x^k} u(t, 0), \tag{16}$$

and set

$$w_0(t, x) = \sum_{n=0}^{\infty} H^n\left[-L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) f_0(t, x)\right], \tag{17}$$

then $w_0(t, x)$ satisfies the following equation

$$w_0(t, x) - H[w_0(t, x)] = -L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) f_0(t, x), \tag{18}$$

and by lemma 2, $w_0(t, x)$ satisfies the inequality (3).

If we set

$$u_0(t, x) = \int_0^t d\tau \int_0^x \frac{(t-\tau)^{j_0-1}(x-y)^{k_0-1}}{(j_0-1)!\,(k_0-1)!} w_0(\tau, y)dy + f_0(t, x), \tag{19}$$

then by (11) and (18) we have

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u_0(t, x) = 0.$$

It is easy to see that $u_0(t, x)$ satisfies the other conditions of the theorem. Since (19) gives

$$\frac{\partial^k}{\partial x^k}(u - u_0(t, x))_{x=0} = 0, \qquad \text{for} \quad 0 \leqq k \leqq k_0 - 1,$$

it follows from lemma 1 that

$$u = u_0(t, x).$$

This completes the proof.

References

[1] Kumano-go, H.: On propergation of regularity in space-variables for the solutions of differential equations with constant coefficients. Proc. Japan Acad., **42**, 204–209 (1966).

[2] Kumano-go, H., and Shinkai, K.: The characterization of differential operators with respect to the characteristic Cauchy problem. Osaka J. of Math., **3**, 155–162 (1966).