30. On Function Spaces over a Topological Semifield

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This paper is devoted to study function spaces, the elements of the space considered are functions from a topological semifield E to the real number field R.

The purpose of this paper is to introduce a topology into the function space and to consider about the property of continuous functions (see [1]).

Let E be a topological semifield and K the positive part of the semifield E.

Every semifield E is a linear topological space over the real number field R (see [2], [3], and [4]). The set K is a convex cone and its closure $\bar{K}$ is also a convex cone. The cone K is called the positive cone of the topological semifield E.

The set $K^{\star}$ of all linear functionals which are non-negative on the positive cone K is called the dual cone, and the set $K^{\star}-K^{\star}$ considering of all differences of elements of $K^{\star}$ is the order dual $E^{\star}$ of E. The cone $K^{\star}$ defines an order on $E^{\star}$ which is called the dual ordering of E.

In particular, the topological semifield E satisfies the equality $K-K=E$. Therefore, the dual ordering is anti-symmetric i.e. if $f \geqslant g$ and $g \geqslant f$ then $f=g$. In this case $f-g$ is zero on each element of K and hence on $E=K-K$.

Next we shall consider the condition under which linear functional is the difference of two positive functionals on the topological semifield E.

Proposition 1. *Let E be a topological semifield. For any two elements $x, y \in E$ we set $\rho(x, y)=|x-y|$. The mapping obtained $\rho: E \times E \rightarrow \bar{K}$ transforms E into a metric space over the semifield E. The weak topology of this metric space coincides with the topology of the semifield E.*

Proposition 2. *Let E be a topological semifield. For any $x \in E$ we set $\|x\|=|x|$. Then E becomes a normed space over the semifield E. The weak topology of this normed space coincides with the topology of the semifield E.*

Let E be a normed space and let E^{*} be the space of all continuous real-valued linear function on E. The norm topology for the adjoint space E^{*} is defined by $\|f\|=\sup\{|f(x)| : \|x\|<1\}$. The topology of positive convergence for E^* is called w^*-topology. A subset F of E^* is called w^*-bdd if for every $x \in E$ the set of all $f(x)$ with f in F is bounded.

Proposition 3. *The space E^* is not complete relative to the w^*-topology unless every linear function on E is continuous.*

Proposition 4 (Alaoglu). *The unit sphere in E^* is compact relative to the w^*-topology. Hence each norm bounded w^*-closed* subset *of E^* is w^*-compact.*

We have next theorem by the preceding discussion.

Theorem 1. *Let E be a topological semifield and K the positive part of E and C be the set of all positve functionals such that* $||f|| \leqslant 1$. *Then each element of E^* is the difference of bounded positive linear functionals if and onlf if $C-C$ is a neighborhood of zero in E^*.*

Proof. First we shall observe the necessity. Since C is w^*-closed subset of unit sphere in E^*, C is w^*-compact and $C-C$ is also w^*-compact. By the condition we have $K^{\star} \cap E^* - K^{\star} \cap E^* = E^*$ and for every f in E^* non-zero scalar m such that $mf \in C-C$. The other hand the unit sphere of E^* is w^*-compact. Therefore there exists a scalar a such that $a(C-C)$ contains the unit sphere of E^*, that is $C-C$ is a neighborhood of 0 in E^*. Conversely, the proof of the sufficiency is obvious.

References

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[3] ——: Metric spaces over a semifield. Work of Univ. Tashkent 191 (1961).

[4] ——: Topological semifield I. Topological Boolean Algebra. Tashkent. U. S. S. R (1963).