23. *A Note on Congruences*

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The object of this note is to give necessary and sufficient conditions when a collection of disjoint non-empty subsets constitute equivalence classes of a congruence (relation) of a universal algebra. This extends previous results by M. Teissier [4] and G. B. Preston [3].

Let $\boldsymbol{A}=(A, O)$ be a universal algebra with operations $O=\{o_i \mid i \in I\}$. Let Σ be the semigroup with identity of functions generated under composition by all unary functions of the forms $o_i(x, a_1, \cdots, a_{n_i-1})$, $o_i(a_0, \cdots, a_{j-1}, x, a_{j+1}, \cdots, a_{n_i-1})$ $(j=1, \cdots, n_i-2)$, and $o_i(a_0, \cdots, a_{n_i-2}, x)$ for some $i \in I$ and some $a_0, a_1, \cdots, a_{n_i-1} \in A$. The class of an equivalence relation θ containing the element a will be denoted by a/θ.

From [2] recall the following

Proposition 1. *A necessary and sufficient condition for an equivalence relation θ on A in a universal algebra $\boldsymbol{A}$ to be a congruence is that if $(x, y) \in \theta$, then $(\sigma(x), \sigma(y)) \in \theta$ for all $x, y \in A$, and $\sigma \in \Sigma$.*

Definition. A subset $S \subseteqq A$ is *intact* under an equivalence relation θ on A if and only if $S \times S \subseteqq \theta$.

Proposition 2. *A subset $S \subseteqq A$ is intact under an equivalence relation θ on A if and only if $S \subseteqq a/\theta$ for some $a \in A$.*

Proof. For self-containment, we shall give a proof. Let S be intact under θ and $a \in S$ so that $S \times S \subseteqq \theta$. If $x \in S$, then $(x, a) \in \theta$ or $x \in a/\theta$. Thus $S \subseteqq a/\theta$. Conversely, suppose $S \subseteqq a/\theta$ for some $a \in A$. If $(x, y) \in S \times S$, so that $x, y \in S$, then $x, y \in a/\theta$ or (x, a), $(y, a) \in \theta$. By reflexivity of θ then (x, a), $(a, y) \in \theta$, and hence $(x, y) \in \theta$ by transitivity. Therefore $S \times S \subseteqq \theta$.

Theorem 3. *Let $\boldsymbol{A}=(A, O)$ be a universal algebra. The minimum congruence under which each member of a collection $\boldsymbol{S}$ of disjoint non-empty subsets of A is intact is the transitive closure $\theta_S = \bigcup_{i=1}^{\infty} \theta^i$ of the relation $\theta=\{(x, y) \mid x, y \in \sigma(T)$ for some $\sigma \in \Sigma$ and some $T \in \boldsymbol{T}\}$, where $\boldsymbol{T}=\boldsymbol{S} \cup \{\{x\} \mid x \in A \setminus \cup \boldsymbol{S}\}$.*

Proof. Observe that the diagonal of A, $\Delta_A \subseteqq \theta \subseteqq \theta_S$ and $\theta^{-1}=\theta$ so that

$$\theta_S^{-1}=\left(\bigcup_{i=1}^{\infty} \theta^i\right)^{-1}=\bigcup_{i=1}^{\infty} (\theta^i)^{-1}=\bigcup_{i=1}^{\infty} (\theta^{-1})^i=\bigcup_{i=1}^{\infty} \theta^i=\theta_S$$

and $\theta_S\theta_S=\left(\bigcup_{i=1}^{\infty}\theta^i\right)\left(\bigcup_{i=1}^{\infty}\theta^i\right)=\bigcup_{i=1}^{\infty}\theta^{i+1}\subseteqq\theta_S$. Thus θ_S is an equivalence relation. Observe that $(x, y)\in\theta_S$ if and only if there exists $a_1, a_2, \cdots, a_m\in A$ $(m\geqq 2)$, and $\sigma_1, \sigma_2, \cdots, \sigma_{m-1}\in\Sigma$ such that $x=a_1$, $y=a_m$, and $a_i, a_{i+1}\in\sigma_i(T)$ $(i=1, \cdots, m-1)$ for some $T\in\boldsymbol{T}$. Hence $\sigma(x)=\sigma(a_1)$, $\sigma(y)=\sigma(a_m)$, and $\sigma(a_i), \sigma(a_{i+1})\in\sigma\sigma_i(T)$ for any $\sigma\in\Sigma$. Since $\sigma\sigma_i\in\Sigma$, then $(\sigma(x), \sigma(y))\in\theta_S\cdot\theta_S$ is therefore a congruence.

Now, let ϕ be any congruence of $\boldsymbol{A}$ under which each member of $\boldsymbol{S}$ is intact. Suppose $(x, y)\in\theta$ so that $x=\sigma(s)$ and $y=\sigma(t)$ for some $s, t\in T$, and some $T\in\boldsymbol{T}$. If $T\in\boldsymbol{S}$, then $(s, t)\in\phi$ and hence $(\sigma(s), \sigma(t))=(x, y)\in\phi$. On the other hand, if $T=\{x\}$, then $(x, y)\in\Delta_A$ and hence $(x, y)\in\phi$. Thus in any case, $\theta\subseteqq\phi$ and therefore $\theta_S\subseteqq\phi$. This shows that θ_S is minimum.

Definition. A subset $S\subseteqq A$ is *full* for a relation θ on A if and only if $(x, y)\in\theta$ and $x\in S$ implies $y\in S$.

Proposition 4. *A subset $S\subseteqq A$ is full for an equivalence relation θ on A if and only if $S=\bigcup_{x\in S}x/\theta$.*

Proof. Suppose that S is full for θ. Then obviously, $S\subseteqq\bigcup_{x\in S}x/\theta$. If $y\in\bigcup_{x\in S}x/\theta$, so that $y\in s/\theta$ or $(s, y)\in\theta$ for some $s\in S$, then $y\in S$. Whence $S=\bigcup_{x\in S}x/\theta$.

Conversely, suppose $S=\bigcup_{x\in S}x/\theta$. Let $x, s\in A$ with $(x, s)\in\theta$ and $s\in S$. Then $x/\theta\cap y/\theta\neq\emptyset$ and hence $x/\theta=s/\theta$ or in other words, $x\in S$.

Theorem 5. *The maximum congruence for which each element of a collection $\boldsymbol{S}$ of non-empty disjoint subsets of A in a universal algebra $\boldsymbol{A}$ is full is the relation θ^S such that $(x, y)\in\theta^S$ if and only if for each $S\in\boldsymbol{S}$, $\sigma(x)\in S$ is equivalent to $\sigma(y)\in S$ for all $\sigma\in\Sigma$. If $\theta=\bigcup_{S\in\boldsymbol{S}}S\times S\cup(A\backslash\cup\boldsymbol{S})\times(A\backslash\cup\boldsymbol{S})$, then $\theta^S=\{(x, y)\mid(\sigma(x), \sigma(y))\in\theta$ for all $\sigma\in\Sigma\}$.*

Proof. Obviously, θ^S is reflexive, symmetric, and transitive and hence, an equivalence relation. Let $\sigma\in\Sigma$ and suppose $(x, y)\in\theta^S$. For each $\sigma'\in\Sigma$ note that $\sigma'\sigma\in\Sigma$. Hence, for all $S\in\boldsymbol{S}$, $\sigma'\sigma(x)\in S$ if and only if $\sigma'\sigma(y)\in S$ for all $\sigma'\in\Sigma$. Thus, $(\sigma(x), \sigma(y))\in\theta^S$ and θ^S is a congruence, by Proposition 1.

Note, if for each $S\in\boldsymbol{S}$, $\sigma(x)\in S$ is equivalent to $\sigma(y)\in S$ for all $\sigma\in\Sigma$, then $(\sigma(x), \sigma(y))\in\theta$ for all $\sigma\in\Sigma$. The converse also holds. Observe that if $(x, y)\in(A\diagdown\cup\boldsymbol{S})\times(A\diagdown\cup\boldsymbol{S})$, then the sentence '$\sigma(x)\in S$ is equivalent to $\sigma(y)\in S$' is trivially true, since both antecedent and consequent are false.

Now, let θ be a congruence of $\boldsymbol{A}$ for which each member of $\boldsymbol{S}$ is full. If $(x, y)\in\theta$, then $(\sigma(x), \sigma(y))\in\theta$ for all $\sigma\in\Sigma$. Hence, for each $S \in \boldsymbol{S}$, $\sigma(x) \in S$ if and only if $\sigma(y) \in S$ for all $\sigma \in \Sigma$. This means that $(x, y) \in \theta^{\boldsymbol{S}}$ and therefore $\theta \subseteqq \theta^{\boldsymbol{S}}$. Thus, $\theta^{\boldsymbol{S}}$ is the maximum congruence of $\boldsymbol{A}$ for which each member of the collection $\boldsymbol{S}$ is full.

Proposition 6. *Let $\boldsymbol{S}$ be a collection of disjoint non-empty subsets of a set A. $\boldsymbol{S}$ are classes of an equivalence relation θ on A if and only if each $S \in \boldsymbol{S}$ is both intact under and full for θ.*

Proof. Observe that a class of an equivalence relation θ on A is always intact under and full for θ. Suppose $S \in \boldsymbol{S}$ is full for an equivalence relation θ on A. Then $S = \bigcup_{x \in S} x/\theta$. If, in addition, S also is intact under θ, then $S \subseteqq s/\theta$ for some $s \in S$. Whence, $S = s/\theta$.

Remarks. Note that any non-empty subset S of A in a universal algebra $\boldsymbol{A}$ is intact under the congruence $A \times A$ of $\boldsymbol{A}$, so that the family $\boldsymbol{L}_{\boldsymbol{S}}$ of all congruences of $\boldsymbol{A}$ under which each member S of $\boldsymbol{S}$ is intact is non-empty. In fact, $\theta_{\boldsymbol{S}} = \bigcap \boldsymbol{L}_{\boldsymbol{S}}$. In particular, $\theta_{\{S\}} = \bigcap \boldsymbol{L}_{\{S\}}$. Moreover, $\theta_{\boldsymbol{S}} = \bigcap \{\theta_{\{S\}} \mid S \in \boldsymbol{S}\}$.

Similarly, any non-empty subset S of A is full for the congruence $\Delta_A = \{(x, x) \mid x \in A\}$ of $\boldsymbol{A}$, so that the family $\boldsymbol{L}^{\boldsymbol{S}}$ of all congruences of $\boldsymbol{A}$ for which each member S of $\boldsymbol{S}$ is full is non-empty and $\theta^{\boldsymbol{S}} = \bigvee \boldsymbol{L}^{\boldsymbol{S}}$. Observe, however, that $\theta^{\boldsymbol{S}}$ is not necessarily equal to $\bigcap \{\theta^{\{S\}} \mid S \in \boldsymbol{S}\}$ where $\theta^{\{S\}} = \bigvee \boldsymbol{L}^{\{S\}}$, although, generally, $\theta^{\boldsymbol{S}} \subseteqq \bigcap \{\theta^{\{S\}} \mid S \in \boldsymbol{S}\}$. On the other hand, observe that $\boldsymbol{L}^{\boldsymbol{S}} = \bigcap \{\boldsymbol{L}^{\{S\}} \mid S \in \boldsymbol{S}\}$.

Theorem 7. *Both $\boldsymbol{L}_{\{S\}}$ and $\boldsymbol{L}^{\{S\}}$ are complete sublattices of the lattice of congruences $\boldsymbol{L}(\boldsymbol{A})$ of a universal algebra $\boldsymbol{A}$ for each subset $S \subseteqq A$.*

Proof. If $\theta_j \in \boldsymbol{L}_{\{S\}}$ for each $j \in J$, so that $S \times S \subseteqq \theta_j$ for each $j \in J$, then $S \times S \subseteqq \bigcap_{j \in J} \theta_j$. Hence $\bigcap_{j \in J} \theta_j \in \boldsymbol{L}_{\{S\}}$. Similarly, if $\theta_j \in \boldsymbol{L}_{\{S\}}$ for each $j \in J$, then the minimal congruence of $\boldsymbol{A}$ that contains their union $\bigcup_{j \in J} \theta_j$ also contains $S \times S$. Whence, $\bigvee_{j \in J} \theta_j \in \boldsymbol{L}_{\{S\}}$.

Let $\theta_j \in \boldsymbol{L}^{\{S\}}$ for each $j \in J$. Then, if $(x, y) \in \bigcap_{j \in J} \theta_j$ and $x \in S$, we have $(x, y) \in \theta_j$ and $x \in S$ for each $j \in J$. This implies that $y \in S$ for all $j \in J$ and therefore $\bigcap_{j \in J} \theta_j \in \boldsymbol{L}^{\{S\}}$. If $(x, y) \in \bigvee_{j \in J} \theta_j$ and $x \in S$, so that for some elements $a_i \in A$, $(a_i, a_{i+1}) \in \theta_{j_i}$ $(i = 1, 2, \cdots, n-1)$ with $x = a_1$ and $y = a_n$, then (by repeated application of the hypothesis) $a_i \in S$ for $i = 1, 2, \cdots, n$. Hence, $y \in S$ and $\bigvee_{j \in J} \theta_j \in \boldsymbol{L}^{\{S\}}$.

Corollary 8. *For each collection $\boldsymbol{S}$ of non-empty disjoint subsets of A in a universal algebra $\boldsymbol{A}$, both $\boldsymbol{L}_{\boldsymbol{S}}$ and $\boldsymbol{L}^{\boldsymbol{S}}$ are complete sublattices of the congruence lattice $\boldsymbol{L}(\boldsymbol{A})$ of the algebra.*

Proof. This result follows from the observation that $\boldsymbol{L}_{\boldsymbol{S}} = \bigcap \{\boldsymbol{L}_{\{S\}} \mid S \in \boldsymbol{S}\}$ and $\boldsymbol{L}^{\boldsymbol{S}} = \{\boldsymbol{L}^{\{S\}} \mid S \in \boldsymbol{S}\}$ and the fact that the intersection of any family of complete lattices is a complete lattice.

Theorem 9. *The following conditions are equivalent for any* *collection* $\boldsymbol{S}$ *of non-empty disjoint subsets of* A *in a universal algebra* $\boldsymbol{A}$:

(1) $\boldsymbol{S}$ *consists of equivalence classes of some congruence of* $\boldsymbol{A}$;

(2) *every* $S \in \boldsymbol{S}$ *is full for the congruence* $\theta_{\boldsymbol{S}}$;

(3) $\theta_{\boldsymbol{S}} \subseteqq \theta^{\boldsymbol{S}}$;

(4) *every* $S \in \boldsymbol{S}$ *is intact under the congruence* $\theta^{\boldsymbol{S}}$;

(5) *if* $\sigma(S_j) \cap S_k \neq \emptyset$, *then* $\sigma(S_j) \subseteqq S_k$ *for all* $\sigma \in \Sigma$ *and* $S_j, S_k \in \boldsymbol{S}$.

Proof. (1) implies (2). Suppose $\boldsymbol{S}$ are equivalence classes of a congruence θ of $\boldsymbol{A}$. This implies that each $S \in \boldsymbol{S}$ is intact under θ and hence $\theta_{\boldsymbol{S}} \subseteqq \theta$ by Theorem 3. Let $S \in \boldsymbol{S}$. If $(x, y) \in \theta_{\boldsymbol{S}}$ and $x \in S$, then also $(x, y) \in \theta$ and $x \in S$, and hence $y \in \boldsymbol{S}$. Condition (2) then follows, since $S \in \boldsymbol{S}$ is arbitrary.

(2) implies (3). The relation $\theta_{\boldsymbol{S}} \subseteqq \theta^{\boldsymbol{S}}$ holds, since $\theta^{\boldsymbol{S}}$ is the maximum congruence of $\boldsymbol{A}$ for which each $S \in \boldsymbol{S}$ is full.

(3) implies (4). For each $S \in \boldsymbol{S}$, it is clear that $S \times S \subseteqq \theta_{\boldsymbol{S}} \subseteqq \theta^{\boldsymbol{S}}$.

(4) implies (1). Each $S \in \boldsymbol{S}$ is naturally full for $\theta^{\boldsymbol{S}}$ and by hypothesis is also intact under $\theta^{\boldsymbol{S}}$. Therefore, each $S \in \boldsymbol{S}$ is an equivalence class of the congruence $\theta^{\boldsymbol{S}}$ (by Proposition 6).

(4) implies (5). By the preceding argument, each $S \in \boldsymbol{S}$ is an equivalence class of the congruence $\theta^{\boldsymbol{S}}$. Thus, for all $x, y \in S_j \in \boldsymbol{S}$, we have $(\sigma(x), \sigma(y)) \in \theta^{\boldsymbol{S}}$ for all $\sigma \in \Sigma$. This means that $\sigma(S_j) \times \sigma(S_j) \subseteqq \theta^{\boldsymbol{S}}$. Since S_j is arbitrary, then each $\sigma(S_j)$ is intact under $\theta^{\boldsymbol{S}}$. Hence, if $\sigma(S_j) \cap S_k \neq \emptyset$ (remembering that S_k is an equivalence class of $\theta^{\boldsymbol{S}}$), then S_k must be precisely the equivalence class of $\theta^{\boldsymbol{S}}$ that contains $\sigma(S_j)$.

(5) implies (4). Let S be an arbitrary element of $\boldsymbol{S}$ and suppose $x, y \in S$ and $\sigma \in \Sigma$. For each $S' \in \boldsymbol{S}$, if $\sigma(x) \in S'$, so that $\sigma(S) \cap S' \neq \emptyset$, then $\sigma(S) \subseteqq S'$ (by hypothesis (5)). Hence $\sigma(x) \in S'$. By symmetry, we also have, if $\sigma(y) \in S'$, then $\sigma(x) \in S'$. Together, we have, $\sigma(x) \in S'$ is equivalent to $\sigma(y) \in S'$ for $S' \in \boldsymbol{S}$. Whence, $(x, y) \in \theta^{\boldsymbol{S}}$. We have thus shown that $\boldsymbol{S} \times \boldsymbol{S} \subseteqq \theta^{\boldsymbol{S}}$ for all $S \in \boldsymbol{S}$ or in other words, each $S \in \boldsymbol{S}$ is intact under $\theta^{\boldsymbol{S}}$.

Theorem 10. *Let* $\boldsymbol{S}$ *be a family of non-empty disjoint subsets of* A. *If the members of* $\boldsymbol{S}$ *are equivalence classes of a congruence of* $\boldsymbol{A}$, *then* $\theta_{\boldsymbol{S}}(\theta^{\boldsymbol{S}})$ *is the minimum (maximum) congruence of* $\boldsymbol{A}$ *which has the family* $\boldsymbol{S}$ *as equvialence classes.*

Proof. Since each $S \in \boldsymbol{S}$ is full for the congruence $\theta^{\boldsymbol{S}}$ (by Theorem 5), then $S = \bigcup_{x \in S} x/\theta^{\boldsymbol{S}}$. On the other hand, since the members of $\boldsymbol{S}$ are equivalence classes of a congruence, then (by Theorem 9), each $S \in \boldsymbol{S}$ is also intact under $\theta^{\boldsymbol{S}}$, that is to say, $S = s/\theta^{\boldsymbol{S}}$ for each $S \in \boldsymbol{S}$ and $s \in S$.

Similarly, by Theorem 9, each $S \in \boldsymbol{S}$ is full for the congruence

θ_S. Since each $S \in \boldsymbol{S}$ is always intact under θ_S (by Theorem 3), then each $S \in \boldsymbol{S}$ is an equivalence class of θ_S (by Proposition 6). Thus, if each $S \in \boldsymbol{S}$ is an equivalence class of a congruence θ of $\boldsymbol{A}$, then each $S \in \boldsymbol{S}$ (by Proposition 6) is both full for and intact under θ. Whence, $\theta_S \subseteqq \theta \subseteqq \theta^S$ (by Theorems 3 and 5).

Theorem 11. *Let **S** be a collection of non-empty disjoint subsets of A. The family of all congruences of a universal algebra **A** for which the members of **S** are equivalence classes constitutes a complete sublattice of the congruence lattice of **A** which has θ_S as its first element and θ^S as its last element.*

Proof. Let the above family be denoted by $\boldsymbol{L}(S)$. Then note that $\boldsymbol{L}(S) = \boldsymbol{L}_S \cap \boldsymbol{L}^S$. By Corollary 8, both $\boldsymbol{L}_S$ and $\boldsymbol{L}^S$ are complete sublattices. Hence, $\boldsymbol{L}(S)$ is also a complete sublattice of $\boldsymbol{L}(A)$ with the prescribed special elements.

Theorem 12. *A collection **S** of non-empty disjoint subsets of A constitutes a family of equivalence classes of precisely one congruence relation θ of a universal algebra **A** if and only if $\theta_S = \theta = \theta^S$.*

The proof follows from the previous Theorem 11.

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