21. *Decomposability of Extension and its Application to Finite Semigroups*[*)]

By Takayuki TAMURA

University of California, Davis, California, U.S.A.

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1. **Introduction.** Let $\mathcal{I}$ be a property preserved by arbitrary homomorphisms, for example, system of identities $\{f_i(x_1, \cdots, x_n) = g_i(x_1, \cdots, x_n); i=1, \cdots, k\}$ where f_i and g_i are words. Let ρ be a congruence on a semigroup S. If S/ρ satisfies $\mathcal{I}$ for all $x_1, \cdots, x_n \in S/\rho$, then ρ is called a $\mathcal{I}$-congruence on S and S/ρ is called a $\mathcal{I}$-homomorphic image of S. It is well known that given $\mathcal{I}$ and S there is a smallest $\mathcal{I}$-congruence ρ_0 on S, that is, if ρ is a $\mathcal{I}$-congruence on S then $\rho_0 \subseteq \rho$. S/ρ_0 is called the greatest $\mathcal{I}$-homomorphic image of S. A semigroup S is called $\mathcal{I}$-indecomposable if the only trivial semigroup is a $\mathcal{I}$-homomorphic image of S. In particular if $\mathcal{I} = \{x^2 = x, xy = yx\}$, ρ is called an s-congruence, S/ρ is an s-homomorphic image of S. The study of finite non-simple s-indecomposable semigroups is reduced to the study of ideal extensions of an s-indecomposable semigroup by an s-indecomposable semigroup with zero. From more general point of view we give a few theorems which are applied to the theory of finite s-indecomposable non-simple semigroups. The terminology in this paper is based on Clifford and Preston's book.[**)]

2. **Basic theorems.** First we introduce some notations. Let ρ be a congruence on a semigroup S. Let H be a subsemigroup of S. $\rho | H$ is the restriction of ρ to H.

Let ξ and η be congruences on S such that $\xi \subseteq \eta$. We define a congruence $\bar{\eta}$ on S/ξ as follows:

$\bar{x}$ denotes the congruence class (modulo ξ) containing x

$$\bar{x}\bar{\eta}\bar{y} \text{ if and only if } x\eta y$$

$\bar{\eta}$ is denoted by $\bar{\eta} = \eta/\xi$.

Let ξ be an equivalence on a set E and A be a subset of E.

A subset $A \cdot \xi$ of E is defined as follows:

$$A \cdot \xi = \{x \in E; x\xi y \text{ for some } y \in A\}.$$

If I is an ideal of a semigroup S and if ξ is a congruence on S, then $I \cdot \xi$ is an ideal of S and $I \subseteq I \cdot \xi$.

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**) A. H. Clifford and G. B. Preston: The algebraic theory of semigroups. Amer. Math. Soc., Providence, R. I. (1961).

Let H be a subsemigroup of S. Let ξ and η be congruences on H and S, respectively. If $\eta | H = \xi$ and if $H \cdot \eta = S$, then η is called a stretched extension of ξ to S, and the homomorphism $S \rightarrow S/\eta$ is called a stretched extension of the homomorphism $H \rightarrow H/\xi$ to S.

Let η be a congruence on S. Then $\eta | (H \cdot \eta)$ is the stretched extension of $\eta | H$ to $H \cdot \eta$ and hence

Lemma 1. $H/(\eta | H) \cong (H \cdot \eta)/(\eta | (H \cdot \eta))$.

Let S/I denote the Rees factor semigroup of S modulo an ideal I.

Proposition 1. *Let I be an ideal of a semigroup and ρ be a congruence on S. Then S/ρ is an ideal extension of $I/(\rho | I)$ by Z where Z is a homomorphic image of S/I.*

Proof. We can prove $S/I \rightarrow S/(I \cdot \rho) \rightarrow (S/\rho)/\{(I \cdot \rho)/(\rho | (I \cdot \rho))\} \cong (S/\rho) /\{I/(I/(\rho | I))\}$ where "$X \rightarrow Y$" denotes "X is homomorphic onto Y." $I/(\rho | I)$ is an ideal of S/ρ.

Lemma 2. *Let I be a $\mathfrak{I}$-indecomposable semigroup, T be a $\mathfrak{I}$-decomposable semigroup with zero and let π be the smallest $\mathfrak{I}$-congruence on T. Let S be an ideal extension of I by T, and let γ denote the Rees-congruence on S modulo I, and ρ the smallest $\mathfrak{I}$-congruence on S. Then*

$$\gamma \subseteq \rho \quad \text{and} \quad \pi \subseteq \rho/\gamma.$$

Proof. Since I is $\mathfrak{I}$-indecomposable, $\gamma \subseteq \rho$ Now ρ/γ is a $\mathfrak{I}$-congruence on S/γ and $S/\gamma \cong T$. Since π is the smallest $\mathfrak{I}$-congruence on S/γ,

$$\pi \subseteq \rho/\gamma.$$

Theorem 2. *Let I be a $\mathfrak{I}$-indecomposable semigroup and T be a $\mathfrak{I}$-indecomposable semigroup with zero. Every ideal extension S of I by T is $\mathfrak{I}$-indecomposable.*

Proof. In Lemma 2, π is the universal relation on T, that is, $\pi = T \times T$ and hence

$$\rho/\gamma = T \times T = (S \times S)/\gamma.$$

It follows that $\rho = S \times S$, since generally if $\gamma \subseteq \rho_1, \gamma \subseteq \rho_2$ and if $\rho_1/\gamma = \rho_2/\gamma$, then $\rho_1 = \rho_2$.

Theorem 3. *Let I be a $\mathfrak{I}$-decomposable semigroup and σ be the smallest $\mathfrak{I}$-congruence on I. Let T be a $\mathfrak{I}$-indecomposable semigroup. If S is an ideal extension of I by T and if S has a $\mathfrak{I}$-congruence ρ such that $\rho | I = \sigma$, then ρ is the smallest $\mathfrak{I}$-congruence on S.*

Proof. Let ρ' be the smallest $\mathfrak{I}$-congruence on S. Since $\rho' \subseteq \rho$, $\rho' | I \subseteq \rho | I = \sigma$ and $\rho' | I$ is a $\mathfrak{I}$-congruence on I, hence

$$\rho' | I = \rho | I = \sigma.$$

Let γ be the Rees-congruence on S modulo I. Let $\rho'\vee\gamma$ denote the congruence generated by ρ' and γ. Clearly $(\rho'\vee\gamma)/\gamma$ is a $\mathfrak{T}$-congruence on S/γ, but S/γ is $\mathfrak{T}$-indecomposable by the assumption. Henceforth

$$(\rho'\vee\gamma)/\gamma=(S/\gamma)\times(S/\gamma)=(S\times S)/\gamma$$

which implies $\rho'\vee\gamma=S\times S$. It can be seen that for each $a\in S$ there is $b\in I$ such that $a\rho' b$. Now we shall prove $\rho\subseteq\rho'$. Let $x\rho y$. By the above remark there are $x', y'\in I$ such that $x\rho' x', y\rho' y'$. Since $\rho'\subseteq\rho$, we have $x'\rho y'$; however, $x'\rho' y'$ because $\rho\mid I=\rho'\mid I$. Therefore we obtain $x\rho' y$. Thus $\rho=\rho'$. This completes the proof.

Theorem 4. *Let I be an ideal of a semigroup S. If ρ is a congruence on I such that I/ρ is a semigroup with a right identity, then there is a stretched extension of ρ to S.*

Proof. Let φ be a right translation of I and suppose that φ is linked with some left translation ψ of I. Since I/ρ has a right identity, there is $a\in S$ such that

$$xa\rho x \text{ for all } x\in S.$$

Now $x\rho y$ implies $x\varphi\rho y\varphi$ since $x\varphi\rho(x\varphi)a=x(a\psi)\rho y(a\psi)=(y\varphi)a\rho y\varphi$. Let $\bar{x}$ denote the equivalence class (modulo ρ) containing x. For φ we define a transformation $\bar{\varphi}$ of I/ρ by

$$\bar{x}\bar{\varphi}=\overline{x\varphi}.$$

Then we can prove that $\bar{\varphi}$ is a right translation of I/ρ and $\varphi\to\bar{\varphi}$ is a homomorphism of S onto the semigroup of all inner right translations of I/ρ, and if $c, b\in I$ then $\bar{\varphi}_c=\bar{\varphi}_b$ implies $c\rho b$.

Corollary. *If f is a homomorphism of I onto a group G, then there is a homomorphism g of S onto G such that g is a stretched extension of f to S. If g is a homomorphism of S onto a group G and if I is an ideal of S, then I is homomorphic onto G under g.*

3. Application. Let S be a finite non-simple s-indecomposable semigroup, and let I be the minimal ideal of S if S has no zero; let I be a 0-minimal ideal of S if S has a zero. In the former case I is a finite simple simigroup; in the latter case, I is either a finite 0-simple semigroup with zero divisors or a null semigroup. T is, of course, an s-indecomposable semigroup with zero.

If S has no zero, a minimal ideal I of S is uniquely determined, but if S has a zero, a 0-minimal ideal of S is not necessarily unique. Instead of 0-minimal ideals we consider the set union M of all 0-minimal ideals. M is also an ideal of S. Let $I_1, I_2, \cdots, I_k$ be all 0-minimal ideals of S. Each one is either a null semigroup or a 0-simple subsemigroup and

$$I_iI_j=\{0\}, \qquad i\neq j.$$

We say that M is the 0-amalgam of $I_1, \cdots, I_k$. (Table II)

By using the results of §2 we can classify all finite s-indecomposable semigroups with proper ideals into 16 classes as the Tables I, II show below.

Let σ be the smallest c-congruence and τ be the smallest i-congruence on S, namely, $c=\{xy=yx\}$, $i=\{x^2=x\}$. A finite simple i-indecomposable semigroup is a proup. Let I be a finite simple semigroup and G^* be the structure group of I. I is c-indecomposable if and only if G^* is c-indecomposable, equivalently, the commutator subgroup K^* of G^* coincides with G^*.[1)]

Table I. Non-simple s-Indecomposable Semigroups Without Zero

I	T	S	S/σ	S/τ	Example I T	Min. Order	Class No.
c-ind group	c-ind	c-ind i-ind			A_5 C_5	64	1.1
	c-dec	c-dec i-ind	nil		A_5 N_2	61	1.2
c-ind i-dec simple	c-ind	c-ind i-dec		rect	$A_5\times R_2$ C_5	124	1.3
	c-dec	c-dec i-dec	nil	rect	$A_5\times R_2$ N_2	121	1.4
c-dec group	c-ind	c-dec i-ind	group		G_2 C_5	6	1.5
	c-dec	c-dec i-ind	u.g.		G_2 N_2	3	1.6
c-dec i-dec simple	c-ind	c-dec i-dec	group	rect	$G_2\times R_2$ C_5	8	1.7
	c-dec	c-dec i-dec	u.g.	rect	$G_2\times R_2 N_2$	5	1.8

Table II. Non 0-simple s-Indecomposable Semigroups With Zero

M	T	S	S/σ	Example M T	Min. Order	Class No.
0-amalagam of 0-simple semigroups	c-ind	c-ind		C_5 C_5	9	2.1
	c-dec	c-dec	nil	C_5 N_2	6	2.2
Null	c-ind	c-ind		N_3 C_5	7	2.3
		c-dec	nil	N_3 C_5	7	2.4
	c-dec	c-dec	nil	N_2 N_2	2	2.5
0-amalgam of 0-simple and Null semigroups	c-ind	c-ind		$N_3{}^\frown C_5$ C_5	11	2.6
		c-dec	nil	$N_3{}^\frown C_5$ C_5	11	2.7
	c-dec	c-dec	nil	$N_2{}^\frown C_5$ N_2	7	2.8

1) T. Tamura: Note on finite simple c-ndecomposable semigroups. Proc. Japan Acad., **35**, 13–15 (1959).

Remarks:

In Table I, "c-ind" means "c-indecomposable"; "c-dec" means "c-decomposable"; "nil" is "nil-semigroup" that is, "unipotent semigroup with zero" ("unipotent" is "with unique idempotent"); "u.g." is "unipotent semigroup containing proper subgroup"; "rect" is "rectangular band"; "null" is "null semigroup" i.e., "semigroup with $xy=0$ for all x, y"; A_5 is the alternative group of degree 5; C_5 is a 0-simple semigroup of order 5 with zero divisor; N_i is a null semigroup of order i, R_2 is a right zero semigroup of order 2; G_2 is a cyclic group of order 2.

In Table II, $N_3 \wedge C_5$ is the 0-amalgam of N_3 and C_5.