43. *An Approach to the Theory of Integration Generated by Positive Linear Functionals and Existence of Minimal Extensions*[*),**)]

By Witold M. BOGDANOWICZ

The Catholic University of America, Washington, D. C., U.S.A.

(Comm. by Kinjirô KUNUGI, M.J.A., March 13, 1967)

A family of sets V of an abstract space X is called a *prering* if V is nonvoid and $A_1, A_2 \in V$ implies $A_1 \cap A_2 \in V$ and there exist disjoint sets $B_1, \cdots, B_k \in V$ such that $A_1 \backslash A_2 = B_1 \cup \cdots \cup B_k$.

A nonnegative function V on a prering is called a *volume* if V is finite-valued and for every countable family $A_t \in V (t \in T)$ of disjoint sets such that $A = \bigcup_T A_t \in V$, we have $v(A) = \sum_T v(A_t)$. Such a triple (X, V, v) will be called a volume space.

In [1] has been presented a direct approach to the theory of Lebesgue-Bochner integration generated by a volume space (X, V, v). The construction of the theory was not based on the theory of measurable functions or on the theory of measure. The construction of the theory of Lebesgue-Bochner measurable functions and of the theory of measure corresponding to this approach has been developed in [3]. In this paper will be presented an approach to the theory of integration generated by a positive linear functional based only on the results of [1].

In § 1 are given equivalent conditions for a linear positive functional on a linear lattice to be a Daniell functional. An extension of the Daniell functional is constructed leading to a positive volume.

In § 2 are given conditions for the integral functional generated by the volume to be an extension of the Daniell functional.

In § 3 are given theorems concerning of the existence of the *smallest extension of a Daniell functional* to an integral functional generated by a volume. It is proven that the volume constructed in § 2 generates the smallest extension, provided that every function $f \in C_0$ is summable with respect to that volume.

In § 4 are given representations of a Daniell functional by means of integral functionals generated by measures. Existence of the smallest measures representing the extensions of the functional are established.

*) This research was partially supported by the NSF grant GP 2565.

) The result contained in Theorem 4 of this paper has been presented to the American Mathematical Society, Notices AMS **13 (1966), p. 89.

In § 5 is given a Baire type characterization of the space of Lebesgue-Bochner summable and measurable functions. Every such a function is equal almost everywhere to a function of the fourth Baire class $C(Y)$ if the class $C_0(Y)$ is defined to consist of sums of functions of the form $f \cdot y$ where $f \in C_0$ and $y \in Y$.

§ 1. Basic extensions of a Daniell functional from the linear lattice. Let C_0 be a linear space of real-valued functions on an abstract set X. Define the following operations $|f|, f \cup g, f \cap g, f^+, f^-$ by means of the formulas $|f|(x) = |f(x)|$, $(f \cup g)(x) = \sup\{f(x), g(x)\}$, $(f \cap g)(x) = \inf\{f(x), g(x)\}$, $f^+(x) = \sup\{f(x), 0\}$, $f^-(x) = \sup\{-f(x), 0\}$ for all $x \in X$.

Proposition 1. *Let C_0 be a linear space of real-valued functions on an abstract set X. Then the following conditions are equivalent:* (a) $|f| \in C_0$ *for all* $f \in C_0$, (b) $f \cup g \in C_0$ *for all* $f, g \in C_0$, (c) $f \cap g \in C_0$ *for all* $f, g \in C_0$, (d) $f^+, f^- \in C_0$ *for all* $f \in C_0$.

A linear space C_0 of real-valued functions on X satisfying one of the conditions of the proposition will be called in this paper a *linear lattice*.

If L is a family of real-valued functions then we define the subfamily L^+ to consist of all nonnegative functions $f \in L$.

Let f, g be two functions on the set X. We shall write $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$. Similarly we define the relation $f \geq g$. We shall say that the sequence of functions f_n is *increasingly (decreasingly) convergent* to f if $f_n \leq f_{n+1}$ ($f_n \geq f_{n+1}$ respectively) and $f_n(x) \to f(x)$ for all $x \in X$. It will be convenient to admit that an increasing unbounded sequence of real numbers is convergent to ∞.

Let J be a real-valued functional on the space C_0. We say that the functional is *positive* if $J(f) \geq 0$ for all $f \in C_0^+$. Notice that if J is a positive linear functional and $f, g \in C_0$ and $f \leq g$ then $Jf \leq Jg$.

The following proposition is basic for constructions of integrals generated by positive linear functionals (compare Bourbaki [4], Loomis [5] p. 30–31, Riesz [6] Chapter 3, Stone [6]).

Proposition 2. *Let C_0 be a linear lattice and J be a linear positive functional on C_0. Then the following conditions are equivalent*

(a) $Jf_n \to 0$ *for every sequence* $f_n \in C_0$ *decreasingly convergent to* 0,

(b) $Jf_n \to Jf$ *for every sequence* $f_n \in C_0$ *increasingly or decreasingly convergent to* $f \in C_0$,

(c) *if* $f, f_n \in C_0^+$ *and* $f = f_1 + f_2 + \cdots$ *then* $Jf = Jf_1 + Jf_2 + \cdots$,

(d) *if* $g, f_n \in C_0$ *and the sequence* f_n *increasingly converges to a function* f *such that* $g \leq f$ *then* $Jg \leq \lim Jf_n$,

(e) *if* $f, f_n \in C_0^+$ *and* $f \leq \sum_{n=1}^{\infty} f_n$ *then* $Jf \leq \sum_{n=1}^{\infty} Jf_n$,

(f) *if* $g_n, f_n \in C_0$ *and* g_n, f_n *increasingly converge to* g *and* f *respectively and* $g \leq f$ *then* $\lim Jg_n \leq \lim Jf_n$.

A linear positive functional J on a linear lattice C_0 will be called a *Daniell functional* if it satisfies one of the conditions (a)—(f) of Proposition 2. The condition (a) and conditions (b), (c), (d), (f) as its consequence were used by Daniell, Riesz, Loomis. The condition (e) as an axiom on the functional J was used by Stone.

Let J be a Daniell functional on a linear lattice C_0. Denote by C the set of all functions f such that there exists a sequence $f_n \in C_0$ increasingly convergent to the function f and such that the sequence Jf_n of numbers is bounded. We allow the function f to take on also the value ∞. Put $Jf = \lim Jf_n$. From the condition (f) of Prop. 2 we see that the expression Jf is well defined, that is it does not depend on the particular choice of the sequence f_n.

Notice that $C_0 \subset C$ and the functional J just defined is an extension of the functional from the space C_0 onto the space C.

Proposition 3. (1) *The set* C *of functions forms a positive cone, that is if* $f, g \in C$ *then* $f+g \in C$, *and if* $t \geq 0$ *and* $f \in C$ *then* $tf \in C$.
(2) *The set* C *is closed under lattice operations, that is if* $f, g \in C$ *then* $f \cup g, f \cap g \in C$.
(3) *The functional* J *is additive, positively homogeneus, and increasing on* C *that is, if* $f, g \in C$ *then* $J(f+g) = Jf + Jg$, *if* $t \geq 0$ *and* $f \in C$ *then* $J(tf) = tJf$, *and if* $f, g \in C$ *and* $f \leq g$ *then* $Jf \leq Jg$.
(4) *If a sequence* $f_n \in C$ *increasingly converges to a function* f *and the sequence* Jf_n *is bounded then* $f \in C$ *and* $Jf_n \to Jf$.

Denote by D the set of all functions f for which there exist finite-valued functions $g_1, g_2 \in C$ such that $f = g_1 - g_2$. Put $Jf = Jg_1 - Jg_2$. From properties of the functional J on the cone C we can easily prove that the above definition is correct. Notice $C_0 \subset C \cap D \subset D$. It is easy to see that the functional J just defined is an extension of the functional from the space C_0 onto the space D.

Proposition 4. (1) *The set* D *of functions is linear.*
(2) *The functional* J *is linear and increasing on* D.
(3) *If* $f, f_n \in D^+$ *and* $f = f_1 + f_2 + \cdots$ *then* $Jf = Jf_1 + Jf_2 + \cdots$.

§ 2. Extension of a Daniell functional to an integral with respect to a volume. Denote by H the family of all sets $Q \subset X$ such that $c_Q \in C$ and there exists a function $f \in C_0$ such that $c_Q \leq f$, where c_Q denotes the characteristic function of the set Q.

Let V be the family of all sets af the form $A = Q_1 \backslash Q_2$ where $Q_1, Q_2 \in H$.

A family of sets is called a *lattice* if it is closed under the operations of finite intersection and finite union.

Theorem 1.

(1) *The family H of sets forms a lattice.*

(2) *The family V of sets forms a prering.*

(3) *If $A \in V$ then $c_A \in D$.*

From (3) of Theorem 1 we get that the function $v(A) = Jc_A$ is well defined for all $A \in V$.

Theorem 2. *The function v is a positive volume on the prering V.*

The triple (X, V, v) will be called the *volume space generated by the functional J.* Let L be the space of real-valued summable functions generated by the volume v. Let $\int$ be the integral functional that is $\int f = \int f\,dv$ for all $f \in L$. We shall write $g \subset f$ if the function f is an *extension* of the function g.

Theorem 3. *The following two conditions are equivalent: $J \subset \int$ and $C_0 \subset L$.*

A related result was obtained by Zaanen [8].

We shall say that the linear lattice C_0 satisfies the *Stone condition* if for every function $f \in C_0$ we have $f \cap 1 \in C_0$. This condition was introduced in [7]. II.

Theorem 4. *If the linear lattice C_0 satisfies the Stone condition then $C_0 \subset L$ and therefore $J \subset \int$.*

§3. Existence of the smallest extension of a Daniell functional to an integral generated by a volume. Let w be a volume and $\int_w$ be the corresponding integral functional defined on the space L_w of all real-valued w-summable functions. If F is a family of such functionals then we shall say that a functional $\int_{w_0} \in F$ is the smallest in F if $\int_{w_0} \subset \int_w$ for all $\int_w \subset F$.

Theorem 5. *The integral functional $\int = \int_v$ is the smallest in the family of all integral functionals $\int_w$ such that $J \subset \int_w$, provided that the linear lattice C_0 is contained in the space L_v of all real-valued v-summable functions.*

§4. Extensions of a Daniell functional to integrals generated by measures and existence of the minimal representing measures. If B is a *sigma-ring* of subsets of the space X and μ is a measure on B then the family

$$W = \{A \in B : \mu(A) < \infty\}$$

is a prering. Let $\bar{\mu}$ be the Lebesgue completion of the measure μ that is the smallest complete measure η such that $\mu \subset \eta$. Denote by w the restriction of the measure μ to the prering W. It is easy to prove that the space $L(w, Y)$ of Lebesgue Bochner summable functions as developed in [1] coinsides with the space $L(\bar{\mu}, Y)$ of Lebesgue-Bochner summable functions obtained by any classical construction and more-over we have $\|\ \|_w = \|\ \|_{\bar{\mu}}$ and $\int f\,dw = \int f\,d\mu$ for all $f \in L(w, Y)$.

In this paper by integration with respect to a measure μ we shall understand the integration generated by the volume w.

Let J be a Daniell functional and (X, V, v) be the volume space generated by J. Let μ_v be the measure generated by v by means of the construction presented in § 3, [3]. Let M be the smallest sigma-ring containing V and μ the restriction of the measure μ_v to M.

Denote by $\int_\mu$ the integral functional defined by the formula $\int_\mu(f) = \int f\,d\mu$ for all $f \in L(\mu, R)$.

Theorem 6. *If $C_0 \subset L = L_v$ then the measure $\mu(\mu_v)$ is the smallest measure (the smallest complete measure, respectively) η such that $J \subset \int_\eta$.*

From this theorem we get

Theorem 7. *The Lebesgue completion $\bar{\mu}$ of the measure μ coinsides with the measure μ_v.*

Indeed from minimality of μ we have $\mu \subset \mu_v$. Since μ_v is complete we have $\mu \subset \bar{\mu} \subset \mu_v$. Using Theorem 8, § 7, [3] we get $\int_v = \int_{\bar{\mu}} = \int_\mu = \int_{\mu_v}$. Thus $J \subset \int_{\bar{\mu}}$. Now from minimality of μ_v we get $\mu_v \subset \bar{\mu}$. This together with the previously proven inclusion yeilds $\bar{\mu} = \mu_v$.

This result can be proven for any volume v directly from Theorem 8, § 7, [3].

In the process of the construction we have proven the following

Theorem 8. *Let J be a Daniell functional on a linear lattice C_0 and v be the volume generated by J. If $C_0 \subset L = L_v$ then $J \subset \int_v = \int_\mu = \int_{\mu_v}$.*

§ 5. **A Baire type characterization of the spaces of Lebesgue-Bochner summable and measurable functions generated by a Daniell functional.** Let $C_0(Y)$ consist of sums of functions of the form $f \cdot y$ where $f \in C_0$ and $y \in Y$, Y being a Banach space. Let the classes $C_n(Y)$ be defined by induction as follows: the class $C_n(Y)$ consists of all *limits under convergence everywhere* of sequences of functions from the class $C_{n-1}(Y)$.

Let $L(Y)$ and $M(Y)$ be the space of Bochner summable functions as in [1] and the space of Bochner measurable functions as defined in [3] generated by the volume v.

Theorem 9. *Let v be the volume generated by a Daniell functional J defined on a linear lattice C_0. If $C_0 \subset L = L_v$ then for every function f from the space $L(Y)$ (or $M(Y)$) there exists a function g from the set $C_4(Y) \cap L(Y)$ (the set $C_4(Y)$ respectively) such that $f(x) = g(x)$ v-almost everywhere on X.*

Compare Bogdanowicz [3], § 6, Theorem 7. We have also

Theorem 10. *Let v be the volume generated by a Daniell functional J defined on a linear lattice C_0. If $C_0 \subset L = L_v$ then for every function f from the space $L(Y)$ (or $M(Y)$) there exists a sequence of functions $f_n \in C_0(Y)$ such that $\| f_n - f \|_v \to 0$ ($f_n \to f$ v-almost everywhere on X respectively).*

A part of the results of this paper will appear in Mathematische Annalen.

References

[1] Bogdanowicz, W. M.: A generalization of the Lebesgue-Bochner-Stieltjes integral and a new approach to the theory of integration. Proc. Nat. Acad., Sc. USA, **53**, 492-498 (1965).

[2] ——: Integral representations of multilinear continuous operators from the space of Lebesgue-Bochner Summable functions into any Banach space, to appear in Trans. Amer. Math. Soc., for announcement of the results see Bull. Amer. Math. Soc., **72**, 317-321 (1966).

[3] ——: An approach to the theory of Lebesgue-Bochner measurable functions and to the theory of measure. Mathem. Annalen, **164**, 251-270 (1966).

[4] Bourbaki, N.: Intégration, Actual. Scient. et Ind., Chap. I-IV. No. 1175 (1952); Chap. V. No. 1244 (1956); Chap. VI. No. 1281 (1959).

[5] Loomis, L. H.: An introduction to abstract harmonic analysis. D. Van Nostrand Co. 1953.

[6] Riesz, F., and Sz.-Nagy, B.: Leçons d'anályse fonctionnelle. Akademiai Kaido, Budapest (1952).

[7] Stone, M. H.: Notes on integration. I, II, III, IV. Proc. Nat. Acad. Sci. USA, **34**, 336-342, 447-455, 483-490 (1948), **35**, 50-58 (1949).

[8] Zaanen, A. C.: A note on the Daniell-Stone integral. Colloque sur l'anlyse fonctionnelle, 63-69 (1961).