42. *Integration with Respect to the Generalized Measure. II*

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The purpose of this part of the present paper is to state a proof of Theorem 1 in [1].

Remark. The proof of Theorem 1 in [1] follows from the propositions (except Propositions 3.1 and 3.2) in section 3 in [1] and therefore this theorem also holds if we replace the assumption for $\mathcal{S}$ in the definition of a structure by the condition stated in the remark in section 3 in [1].

Denote by $\mathcal{G}_1$ the perfection of $\mathcal{G}$ and by $\mathcal{G}_2$ the perfection of the closure $\bar{\mathcal{G}}_1$ of $\mathcal{G}_1$ in $\mathcal{F}$.

Lemma 1. *The integral closure $\tilde{\mathcal{G}}$ of $\mathcal{G}$ is the $\mathcal{F}$-completion of $\mathcal{G}_2$.*

Proof. Let $\mathcal{G}_3$ be the $\mathcal{F}$-completion of $\mathcal{G}_2$. Then Proposition 3.10 [1] implies that $\mathcal{G}_3$ is the $\mathcal{F}$-completion of $\bar{\mathcal{G}}_1$. Hence it follows from Proposition 3.17 [1] that $\mathcal{G}_3$ is closed and therefore $\mathcal{G}_3$ is i-closed. To prove that $\mathcal{G}\subset\mathcal{G}_3$, let us consider the $\mathcal{F}$-completion $\mathcal{G}_4$ of $\mathcal{G}_1$. Then Proposition 3.10 [1] implies that $\mathcal{G}\subset\mathcal{G}_4$ and the formula $\mathcal{G}_1\subset\bar{\mathcal{G}}_1$ implies that $\mathcal{G}_4\subset\mathcal{G}_3$. Thus we have $\mathcal{G}\subset\mathcal{G}_3$. It is easily verified that $\mathcal{G}_3$ is the smallest of i-closed subgroups of $\mathcal{F}$ containing $\mathcal{G}$. This proves the lemma.

Let I be the perfection of $\mathcal{I}$ and let I_X be the restriction of I on $X\mathcal{G}_1$ for each $X\in\mathcal{S}$. Then I_X is a continuous homomorphism of $X\mathcal{G}_1$ into J for each $X\in\mathcal{S}$.

Lemma 2. *I_X is uniquely extended to a continuous homomorphism $\bar{I}_X$ of $X\bar{\mathcal{G}}_1$ into J for each $X\in\mathcal{S}$.*

Proof. From the continuity of X, it follows that $X\bar{\mathcal{G}}_1\subset\overline{X\mathcal{G}_1}$ and therefore that $X\mathcal{G}_1$ is dense in $X\bar{\mathcal{G}}_1$. Since J is Hausdorff and complete, this lemma follows from Bourbaki.[1)]

Considering the map $\bar{I}_X$ in Lemma 2, we have

Lemma 3. *There uniquely exists an integral map $\bar{I}$ with respect to $(\mathcal{S}, \mathcal{G}_2, J)$ such that the restriction of $\bar{I}$ on $X\bar{\mathcal{G}}_1$ coincides with $\bar{I}_X$ for each $X\in\mathcal{S}$.*

Proof. Let us prove that $\bar{I}_X(f)=\bar{I}_Y(f)$ for $X, Y\in\mathcal{S}$, and

1) [2] chap. III. Groupes Topologiques, §3, no 3, Proposition 5.

$f\in(X\bar{\mathcal{G}}_1)\cap(Y\bar{\mathcal{G}}_1)$. For $Z\in\mathcal{S}$ such that $ZX=X, ZY=Y$, it follows from Proposition 3.3 [1] that $Z\bar{\mathcal{G}}_1\supset X\bar{\mathcal{G}}_1$ and $Z\bar{\mathcal{G}}_1\supset Y\bar{\mathcal{G}}_1$. Since the restriction $\bar{I}_X'$ of $\bar{I}_Z$ on $X\bar{\mathcal{G}}_1$ is a continuous homomorphism which is an extension of I_X, the uniqueness of such an extension implies that $\bar{I}_X'=\bar{I}_X$. Similarly the restriction $\bar{I}_Y'$ of $\bar{I}_Z$ on $Y\bar{\mathcal{G}}_1$ coincides with $\bar{I}_Y$. Hence $\bar{I}_X(f)=\bar{I}_X'(f)=\bar{I}_Z(f)=\bar{I}_Y'(f)=\bar{I}_Y(f)$.

Thus we can define a map $\bar{I}$ of $\mathcal{G}_2=\bigcup_{X\in\mathcal{S}}(X\bar{\mathcal{G}}_1)$ into J such that $\bar{I}(f)=\bar{I}_X(f)$ for $X\in\mathcal{S}$ and $f\in X\bar{\mathcal{G}}_1$. Since $X\mathcal{G}_2\subset X\bar{\mathcal{G}}_1$, which follows from $\mathcal{G}_2\subset\bar{\mathcal{G}}_1$, the restriction of $\bar{I}$ on $X\mathcal{G}_2$ is the restriction of $\bar{I}_X$ and consequently is a continuous homomorphism for each $X\in\mathcal{S}$. Thus it is proved that there exists an integral map $\bar{I}$ with respect to $(\mathcal{S}, \mathcal{G}_2, J)$ satisfying the condition stated in the lemma. The uniqueness is obvious and this completes the proof of the lemma.

Proof of Theorem 1 in [1]. Let $\tilde{\mathcal{J}}$ be the $\mathcal{F}$-completion of the integral map $\bar{I}$ in Lemma 3. It follows from Lema 1 that $\tilde{\mathcal{J}}$ is an integral with respect to $(\mathcal{S},\tilde{\mathcal{G}},J)$. Let us prove that $\tilde{\mathcal{J}}(X, g)=\mathcal{J}(X, g)$ for each $X\in\mathcal{S}$ and $g\in\mathcal{G}$. Since $Xg\in X\mathcal{G}=X(X\mathcal{G})\subset X\mathcal{G}_1\subset X\bar{\mathcal{G}}_1$, we have $\tilde{\mathcal{J}}(X, g)=\bar{I}(Xg)=\bar{I}_X(Xg)=I_X(Xg)=I(Xg)=\mathcal{J}(X, g)$. Thus it is proved that $\mathcal{J}$ is extended to an integral $\tilde{\mathcal{J}}$ with respect to $(\mathcal{S}, \tilde{\mathcal{G}}, J)$.

The uniqueness of such an extension is proved as follows. Let $\tilde{\mathcal{J}}'$ be such an extension of $\mathcal{J}$. Denote by $\bar{I}'$ the perfection of $\tilde{\mathcal{J}}'$ and by I' the restriction of $\bar{I}'$ on $\mathcal{G}_1$. Then it is easily verified that I' and $\bar{I}'$ coincides with I and $\bar{I}$, respectively. Hence Proposition 3.15 [1] implies that $\tilde{\mathcal{J}}'=\tilde{\mathcal{J}}$ and thus Theorem 1 is proved.

References

[1] M. Takahashi: Integration with respect to the generalized measure. I. Proc. Japan Acad., **43**, 178–183 (1967).

[2] N. Bourbaki: Topologie générale (Elément de Mathématique. III. 3e éd.). Hermann, Paris (1960).