41. *Integration with Respect to the Generalized Measure. I*

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1. Introduction. In this paper, we are going to deal with the integration theory with respect to the *topological-additive-group-valued measure* [1].

Let M be a set and $\mathcal{S}$ a ring of subsets of M ($\mathcal{S}$ is a ring in the algebraic sense,[1] of which each element is an idempotent). Let μ be a *measure* [1] defined on $\mathcal{S}$ taking values in a topological additive group G.

Let K be a topological additive group and let $\mathcal{F}$ be the additive group of all K-valued functions defined on M (the sum of two functions in $\mathcal{F}$ is defined in the usual way).

For $X\in\mathcal{S}$ and $f\in\mathcal{F}$, let us denote by Xf the function in $\mathcal{F}$ such that

$$(Xf)(x)=\begin{cases} f(x) & \text{if} \quad x\in X, \\ 0 & \text{if} \quad x\in M-X. \end{cases}$$

Then each $X\in\mathcal{S}$ operates as a homomorphism on the group $\mathcal{F}$. We further assume that $\mathcal{F}$ is a topological group with some topology such that each $X\in\mathcal{S}$ operates as a continuous map on $\mathcal{F}$.

Let J be a topological additive group and suppose that a map of $G\times K$ into J, denoting by $g\cdot k$ the image of (g, k), $g\in G$, $k\in K$, is defined, satisfying the conditions:

1) $(g+g')\cdot k=g\cdot k+g'\cdot k$,
2) $g\cdot(k+k')=g\cdot k+g\cdot k'$,

for each $g, g'\in G$ and $k, k'\in K$.

As an illustration, suppose that M is the real line and $G=K=J$ is the topological ring of all real numbers. Let $\mathcal{S}$ be the *pseudo-σ-ring* [1] of measure[2]-finite Lebesgue measurable sets and μ the Lebesgue measure on $\mathcal{S}$ (strictly, its restriction on $\mathcal{S}$). Now we can consider $\mathcal{F}$ as a topological additive group introducing the topology in such a way that a sequence of functions in $\mathcal{F}$ converges in the space $\mathcal{F}$ if and only if the sequence uniformely converges as a functional sequence. Then, each $X\in\mathcal{S}$ operates as a continuous homomorphism of $\mathcal{F}$ into itself.

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1) $X+Y=(X-Y)\cup(Y-X)$, $XY=X\cap Y$ for each $X, Y\in\mathcal{S}$.

2) Lebesgue measure.

Let $\mathcal{G}$ be the set of all bounded measurable functions in $\mathcal{F}$. Then $\mathcal{G}$ is an $\mathcal{S}$-*invariant*[3] subgroup of $\mathcal{F}$.

For $X \in \mathcal{S}$, $g \in \mathcal{G}$, write

$$\int_X g d\mu = \mathcal{J}(X, g).$$

Then $\mathcal{J}$ is a map of $\mathcal{S} \times \mathcal{G}$ into J with the properties:

(*) The map $\mathcal{J} = \mathcal{J}(X, g)$ is a continuous homomorphism of $\mathcal{G}$ into J with respect to g for any fixed X.

(**) $\mathcal{J}(XY, g) = \mathcal{J}(X, Yg)$ for each of $X, Y \in \mathcal{S}$, and $g \in \mathcal{G}$.

(***) If $g(x) = k$ for every $x \in M$, then $\mathcal{J}(X, g) = \mu(X) \cdot k$.

It will be known that the Lebesgue integral $\int_X g d\mu = \mathcal{J}(X, g)$ of $g \in \mathcal{G}$ over $X \in \mathcal{S}$ is characterized by these three properties.

In general, we shall define an integral $\mathcal{J}$ as a map of $\mathcal{S} \times \mathcal{G}$, $\mathcal{G}$ being an $\mathcal{S}$-invariant subgroup of $\mathcal{F}$, into J satisfying the conditions above.

In part I in this paper, we shall deal in some abstract way with the process of extending a primitive integral to an integral which has a wider class of 'integrable' functions.

It may be noted that for the purpose of constructing an integral the countable additivity of the measure μ is of no use. This property is used to prove that $\mathcal{J}\left(\bigcup_{i=1}^{\infty} X_i, g\right) = \sum_{i=1}^{\infty} \mathcal{J}(X_i, g)$ for some X_i's in $\mathcal{S}$ and $g \in \mathcal{G}$.

2. An abstract integral and an extension theorem. Let $\mathcal{S}$ be a ring (in the algebraic sense), of which each element is an idempotent. Let $\mathcal{F}$ be a topological additive group and assume that each $X \in \mathcal{S}$ operates as a continuous homomorphism of $\mathcal{F}$ into itself satisfying the conditions:

1) $(X+Y)f = Xf + Yf$ if $XY = 0$,

2) $(XY)f = X(Yf)$,

for each $X, Y \in \mathcal{S}$ and $f \in \mathcal{F}$. Then for a topological additive group J we shall call the triplet $(\mathcal{S}, \mathcal{F}, J)$ an *abstract integral structure* or briefly a *structure*. If $(\mathcal{S}, \mathcal{F}, J)$ is a structure, for any $\mathcal{S}$-invariant subgroup $\mathcal{G}$ of $\mathcal{F}$, $(\mathcal{S}, \mathcal{G}, J)$ is a structure.

Let $(\mathcal{S}, \mathcal{F}, J)$ be a structure. A closed subgroup $\mathcal{G}$ of $\mathcal{F}$ is called an *i-closed* subgroup of $\mathcal{F}$ if it holds that $\mathcal{G} = \{g \mid g \in \mathcal{F}, Xg \in \mathcal{G}$ for any $X \in \mathcal{S}\}$. If $\mathcal{G}$ is an *i*-closed subgroup of $\mathcal{F}$, then $\mathcal{G}$ is an $\mathcal{S}$-invariant subgroup and consequently $(\mathcal{S}, \mathcal{G}, J)$ is a structure.

Proposition 2.1. *Let $(\mathcal{S}, \mathcal{F}, J)$ be a structure and $\mathcal{A}$ a subset of $\mathcal{F}$. Then there is the smallest i-closed subgroup $\mathcal{G}$ of $\mathcal{F}$ containing $\mathcal{A}$.*

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3) $X\mathcal{G} \subset \mathcal{G}$ for each $X \in \mathcal{S}$.

Proof. Let Γ be the class of all i-closed subgroups of $\mathcal{F}$ containing $\mathcal{A}$. Since $\mathcal{F}\in\Gamma$, it is sufficient to show that $\bigcap_{\mathcal{H}\in\Gamma}\mathcal{H}\in\Gamma$ and this is easily seen.

The subgroup $\mathcal{G}$ of $\mathcal{F}$ in Proposition 2.1 is called the *integral closure* of $\mathcal{A}$ in $\mathcal{F}$.[4]

Let $(\mathcal{S},\mathcal{F},J)$ be a structure. A map $\mathcal{J}$ of $\mathcal{S}\times\mathcal{F}$ into J is called an *abstract integral* or briefly an *integral* with respect to $(\mathcal{S},\mathcal{F},J)$ if it satisfies the conditions:

(*) The map $\mathcal{J}=\mathcal{J}(X,f)$ is a continuous homomorphism of $\mathcal{F}$ into J with respect to f for any fixed X.

(**) $\mathcal{J}(XY,f)=\mathcal{J}(X,Yf)$ for each $X,Y\in\mathcal{S}$ and $f\in\mathcal{F}$.

We shall state the main theorem with respect to the extension of an abstract integral, which will be proved in part II of this paper.

Theorem 1. *Let $(\mathcal{S},\mathcal{F},J)$ be a structure and assume that J is a Hausdorff, complete group. Let $\mathcal{G}$ be an $\mathcal{S}$-invariant subgroup of $\mathcal{F}$ and let $\mathcal{J}$ be an integral with respect to the structure $(\mathcal{S},\mathcal{G},J)$. Then the integral $\mathcal{J}$ is uniquely extended to an integral $\tilde{\mathcal{J}}$ with respect to the structure $(\mathcal{S},\tilde{\mathcal{G}},J)$, where $\tilde{\mathcal{G}}$ is the integral closure of $\mathcal{G}$ in $\mathcal{F}$.*

3. Integral maps and some propositions.

Assumption. *In this section we assume that $(\mathcal{S},\mathcal{F},J)$ is a structure and $\mathcal{G}$ is an $\mathcal{S}$-invariant subgroup P of $\mathcal{F}$.*

Proposition 3.1. *For each $X,Y\in\mathcal{S}$, it holds that*

1) $XY=YX$,

2) $X+X=0$

3) $ZX=X, ZY=Y$ *for some* $Z\in\mathcal{S}$.

Proof. The formula $X+Y=(X+Y)^2=X^2+XY+YX+Y^2=X+XY+YX+Y$ implies that $XY+YX=0$. Replacing Y by X we have $X+X=X^2+X^2=0$, which proves 2). Further we have $XY=XY+(XY+YX)=(XY+XY)+YX=YX$, proving 1). Putting $Z=X+Y+XY$, we have 3).

Proposition 3.2. *If $\mathcal{J}$ is an integral with respect to $(\mathcal{S},\mathcal{F},J)$, then*

$\mathcal{J}(X+Y,f)=\mathcal{J}(X,f)+\mathcal{J}(Y,f)$ *if $XY=0$, for each $X,Y\in\mathcal{S}$, and $f\in\mathcal{F}$.*

Proof. $\mathcal{J}(X+Y,f)=\mathcal{J}((X+Y)^2,f)=\mathcal{J}(X+Y,(X+Y)f)=\mathcal{J}(X+Y,Xf+Yf)=\mathcal{J}(X+Y,Xf)+\mathcal{J}(X+Y,Yf)=\mathcal{J}((X+Y)X,f)+\mathcal{J}((X+Y)Y,f)=\mathcal{J}(X,f)+\mathcal{J}(Y,f)$.

4) Let $\mathcal{F}$ be the group stated in the example in section 1 and let $\mathcal{K}$ be the set of all constant valued functions in $\mathcal{F}$. Then the subgroup $\mathcal{G}$ of $\mathcal{F}$ in that example is contained in the integral closure $\tilde{\mathcal{K}}$ of $\mathcal{K}$ in $\mathcal{F}$. In fact $\tilde{\mathcal{K}}$ is the class of functions f in $\mathcal{F}$ such that $Xf\in\mathcal{G}$ for any $X\in\mathcal{S}$ (cf. Propostion 3. 17).

Remark. To obtain the following propositions, we can replace the assumption that $\mathcal{S}$ is a ring, of which each element is an idempotent, by a weaker one: $\mathcal{S}$ is a set and for each $X, Y \in \mathcal{S}$, the product XY is defined as an element of $\mathcal{S}$, satisfying the conditions:

a) $X^2 = X$,

b) $XY = YX$,

c) $ZX = X, ZY = Y$ for some $Z \in \mathcal{S}$,

for each $X, Y \in \mathcal{S}$. Consequently the condition '$(X+Y)f = Xf + Yf$ for $X, Y \in \mathcal{S}$ such that $XY = 0$ and for any $f \in \mathcal{F}$' may be omitted.

Proposition 3.3. *If $X, Y \in \mathcal{S}$, then*

1) *$X\mathcal{G}$ is an $\mathcal{S}$-invariant subgroup of $\mathcal{F}$,*

2) *$X\mathcal{G} \subset Y\mathcal{G}$ if $X = XY$.*

Proposition 3.4. *$\bar{\mathcal{G}}$[5] is an $\mathcal{S}$-invariant subgroup of $\mathcal{F}$.*

Proof. The continuity of $X \in \mathcal{S}$ and the $\mathcal{S}$-invariance of $\mathcal{G}$ implies the $\mathcal{S}$-invariance of $\bar{\mathcal{G}}$ as $X\bar{\mathcal{G}} \subset \overline{X\mathcal{G}} \subset \bar{\mathcal{G}}$.

Proposition 3.5. *The following three conditions are mutually equivalent.*

1) *For each $g \in G$, there exists $X \in \mathcal{S}$ such that $Xg = g$,*

2) $\mathcal{G} \subset \bigcup_{X \in \mathcal{S}} (X\mathcal{G})$,

3) $\mathcal{G} = \bigcup_{X \in \mathcal{S}} (X\mathcal{G})$.

If an $\mathcal{S}$-invariant subgroup $\mathcal{G}$ of $\mathcal{F}$ satisfies the mutually equivalent conditions, in Proposition 3.5, then we shall say that $\mathcal{G}$ is *perfect*.

Proposition 3.6. *If we put $\mathcal{G}' = \bigcup_{X \in \mathcal{S}} (X\mathcal{G})$, $\mathcal{G}'$ is the largest perfect subgroup of $\mathcal{G}$.*

Proof. To prove that $\mathcal{G}'$ is a subgroup of $\mathcal{G}$, since $\mathcal{G}' \subset \mathcal{G}$, it is sufficient to show that $Xf - Yg \in \mathcal{G}'$ for each $X, Y \in \mathcal{S}$ and $f, g \in \mathcal{G}$. The $\mathcal{S}$-invariance of $\mathcal{G}$ implies that $Xf - Yg \in \mathcal{G}$. For $Z \in \mathcal{S}$ such that $ZX = X, ZY = Y$, we have $Xf - Yg = ZXf - ZYg = Z(Xf - Yg) \in Z\mathcal{G} \subset \mathcal{G}'$. The remaining part is easily verified.

The largest perfect subgroup $\mathcal{G}'$ of $\mathcal{G}$ in Proposition 3.6 is called the *perfection* of $\mathcal{G}$.

Proposition 3.7. *The following three conditions are mutually equivalent.*

1) *If $f \in \mathcal{F}$ and if $Xf \in \mathcal{G}$ for each $X \in \mathcal{S}$, then $f \in \mathcal{G}$,*

2) $\mathcal{G} \supset \bigcap_{X \in \mathcal{S}} (X^{-1}\mathcal{G})$,[6]

3) $\mathcal{G} = \bigcap_{X \in \mathcal{S}} (X^{-1}\mathcal{G})$.

If an $\mathcal{S}$-invariant subgroup $\mathcal{G}$ of $\mathcal{F}$ satisfies the mutually equiva-

5) $\bar{\mathcal{G}}$ means the closure of $\mathcal{G}$ in $\mathcal{F}$ in the topological sense.

6) X^{-1} means the inverse map of the map X of $\mathcal{F}$ into itself.

lent conditions in Proposition 3.7, then we shall say that $\mathcal{G}$ is $\mathcal{F}$-*complete*. A necessary and sufficient condition that an $\mathcal{S}$-invariant subgroup $\mathcal{G}$ of $\mathcal{F}$ be i-closed is that $\mathcal{G}$ be closed in $\mathcal{F}$ and $\mathcal{F}$-complete.

Proposition 3.8. *If we put* $\mathcal{G}''=\bigcap_{X\in\mathcal{S}}(X^{-1}\mathcal{G})$, $\mathcal{G}''$ *is the smallest* $\mathcal{F}$-*complete subgroup of* $\mathcal{F}$ *containing* $\mathcal{G}$.

The smallest $\mathcal{F}$-complete subgroup $\mathcal{G}''$ of $\mathcal{F}$ containing $\mathcal{G}$ in Proposition 3.8 is called the $\mathcal{F}$-*completion* of $\mathcal{G}$.

Proposition 3.9. *The perfection* $\mathcal{G}'$ *of the* $\mathcal{F}$-*completion of* $\mathcal{G}$ *coincides with the perfection of* $\mathcal{G}$. *If, in particular,* $\mathcal{G}$ *is perfect, then* $\mathcal{G}'=\mathcal{G}$.

Proposition 3.10. *The* $\mathcal{F}$-*completion* $\mathcal{G}''$ *of the perfection of* $\mathcal{G}$ *coincides with the* $\mathcal{F}$-*completion of* $\mathcal{G}$. *If, in particular,* $\mathcal{G}$ *is* $\mathcal{F}$-*complete, then* $\mathcal{G}''=\mathcal{G}$.

A map I of $\mathcal{F}$ into J is called an *integral map* with respect to the structure $(\mathcal{S},\mathcal{F},J)$ if, for each $X\in\mathcal{S}$, the restriction I_X of I on the group $X\mathcal{F}$ is a continuous homomorphism.

Proposition 3.11. *If* I *is an integral map with respect to* $(\mathcal{S},\mathcal{G},J)$ *and if* $\mathcal{G}$ *is perfect, then* I *is a homomorphism of* $\mathcal{G}$ *into* J.

Proof. Our assertion is that $I(g+h)=I(g)+I(h)$ for each $g,h\in\mathcal{G}$. The perfectness of $\mathcal{G}$ implies that there exist $X,Y\in\mathcal{S}$ such that $g=Xg$, $h=Yh$. For $Z\in\mathcal{S}$ such that $ZX=X$, $ZY=Y$, we have $g=Xg=ZXg=Zg\in Z\mathcal{G}$ and similarly $h\in Z\mathcal{G}$. Since the restriction I_Z of I on $Z\mathcal{G}$ is a homomorphism, it follows that $I(g+h)=I_Z(g+h)=I_Z(g)+I_Z(h)=I(g)+I(h)$.

Proposition 3.12. *Let* $\mathcal{J}$ *be an integral with respect to* $(\mathcal{S},\mathcal{G},J)$ *and let* $\mathcal{G}'$ *be the perfection of* $\mathcal{G}$. *Then there uniquely exists an integral map* I *with respect to* $(\mathcal{S},\mathcal{G}',J)$ *such that*

$$I(Xg)=\mathcal{J}(X,g)\ \textit{for}\ X\in\mathcal{S}\ \textit{and}\ g\in\mathcal{G}.$$

Proof. Let us prove that $\mathcal{J}(X,g)=\mathcal{J}(Y,h)$ for $X,Y\in\mathcal{S}$ and $g,h\in\mathcal{G}$ such that $Xg=Yh$. Putting $Xg=Yh=f$, we have $f=Yh=Y^2h=Yf$. Hence $\mathcal{J}(X,g)=\mathcal{J}(X^2,g)=\mathcal{J}(X,Xg)=\mathcal{J}(X,f)=\mathcal{J}(X,Yf)=\mathcal{J}(XY,f)$. Similarly we have $\mathcal{J}(Y,h)=\mathcal{J}(XY,f)$ and this implies that $\mathcal{J}(X,g)=\mathcal{J}(Y,h)$. Thus we can define a map I of $\mathcal{G}'$ into J such that $I(Xg)=\mathcal{J}(X,g)$ for each $X\in\mathcal{S}$ and $g\in\mathcal{G}$. It is easily seen that I is an integral map required and the uniqueness of I is obvious.

The integral map I in Proposition 3.12 is called the *perfection* of $\mathcal{J}$.

Proposition 3.13. *Let* I *be an integral map with respect to* $(\mathcal{S},\mathcal{G},J)$ *and let* $\mathcal{G}''$ *be the* $\mathcal{F}$-*completion of* $\mathcal{G}$. *Then there uniquely exists an integral* $\mathcal{J}$ *with respect to* $(\mathcal{S},\mathcal{G}'',J)$ *such that*

$$\mathcal{J}(X,g)=I(Xg)\ \textit{for}\ X\in\mathcal{S}\ \textit{and}\ g\in\mathcal{G}''.$$

The integral $\mathcal{J}$ in Proposition 3.13 is called the $\mathcal{F}$-*completion* of I.

Proposition 3.14. *If I is an integral map with respect to $(\mathcal{S}, \mathcal{G}, J)$, then the perfection I' of the $\mathcal{F}$-completion of I is the restriction of I. If, in particular, $\mathcal{G}$ is perfect, then I' coincides with I.*

Proposition 3.15. *If $\mathcal{I}$ is an integral with respect to $(\mathcal{S}, \mathcal{G}, J)$, then the $\mathcal{F}$-completion $\mathcal{I}''$ of the perfection of $\mathcal{I}$ is an extension of $\mathcal{I}$. If, in particular, $\mathcal{G}$ is $\mathcal{F}$-complete, then $\mathcal{I}'$ coincides with $\mathcal{I}$.*

Proposition 3.16. *If $\mathcal{I}$ is an integral with respect to $(\mathcal{S}, \mathcal{G}, J)$ and if $\mathcal{G}''$ is the $\mathcal{F}$-completion of $\mathcal{G}$, then $\mathcal{I}$ is uniquely extended to an integral $\mathcal{I}''$ with respect to $(\mathcal{S}, \mathcal{G}'', J)$.*

Proof. This follows immediately from Propositions 3.15 and 3.10.

The integral $\mathcal{I}''$ in Proposition 3.16 is called the *$\mathcal{F}$-completion* of $\mathcal{I}$.

Proposition 3.17. *If $\mathcal{G}$ is closed in $\mathcal{F}$, then the $\mathcal{F}$-completion $\mathcal{G}''$ of $\mathcal{G}$ is closed in $\mathcal{F}$.*

Proof. Since each $X \in \mathcal{S}$ is continuous, we have $\overline{\mathcal{G}''} = \overline{\bigcap_{X \in \mathcal{S}} (X^{-1}\mathcal{G})} \subset \bigcap_{X \in \mathcal{S}} (\overline{X^{-1}\mathcal{G}}) \subset \bigcap_{X \in \mathcal{S}} (X^{-1}\bar{\mathcal{G}}) = \bigcap_{X \in \mathcal{S}} (X^{-1}\mathcal{G}) = \mathcal{G}''$, which proves the proposition.

Corollary. *The integral closure of $\mathcal{G}$ in $\mathcal{F}$ is the $\mathcal{F}$-completion $\overline{\mathcal{G}''}$ of the closure $\bar{\mathcal{G}}$ of $\mathcal{G}$ in $\mathcal{F}$.*

References

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