40. On Pairs of Very-Close Formal Systems

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While we were examining mutual relations between formal systems, we were rather astonished by finding out that there exists a pair of distinct formal systems[1] $\mathbf{M}_1$ and $\mathbf{M}_2$ and another formal system $\mathbf{N}$ stronger than $\mathbf{M}_1$ and $\mathbf{M}_2$ and satisfying the following condition $\mathfrak{C}(\mathbf{M}_1, \mathbf{M}_2, \mathbf{N})$: *For any finite number of propositions* $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$, *the system* $\mathbf{M}_1[\mathfrak{x}_1, \cdots, \mathfrak{x}_n]$ *is equivalent to* $\mathbf{N}$ *if and only if* $\mathbf{M}_2[\mathfrak{x}_1, \cdots, \mathfrak{x}_n]$ *is so, where* $\mathbf{M}_i[\mathfrak{x}_1, \cdots, \mathfrak{x}_n]$ *denotes the formal system stronger than* $\mathbf{M}_i$ *by the axioms* $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ $(i=1, 2)$.

Any pair of formal systems $\mathbf{M}_1$ and $\mathbf{M}_2$ is called *very-close* if and only if they have such a formal system $\mathbf{N}$ that satisfies $\mathfrak{C}(\mathbf{M}_1, \mathbf{M}_2, \mathbf{N})$. Restricting to formal systems each being stronger than a certain formal system standing on a logic admitting inferences of the implication logic[2] by a finite number of axioms, we can find out a necessary and sufficient condition for any pair of formal systems $\mathbf{M}_1$ and $\mathbf{M}_2$ to be *very-close*. This short note is to exhibit a theorem which gives the condition.

The condition can be stated very simply in the case where the logic has *conjunction* as its logical constant. In this case, any number of axioms can be unified into a single axiom. Here we have: *Two formal systems* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are very-close if and only if we can find out a formal system* $\mathbf{F}$ *and a pair of propositions* $\mathfrak{p}$ *and* $\mathfrak{q}$ *such that* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *lie between* $\mathbf{F}[\mathfrak{P}]$ *and* $\mathbf{F}[\mathfrak{p}]$, *where* $\mathfrak{P}$ *stands for* $(\mathfrak{p}\to\mathfrak{q})\to\mathfrak{p}$.

Taking $\mathfrak{p}$ and $\mathfrak{q}$ as $\mathfrak{p}_1\wedge \cdots \wedge\mathfrak{p}_s$ and $\mathfrak{q}_1\wedge \cdots \wedge\mathfrak{q}_t$, respectively, we can interpret the above theorem even in the case where we do not assume *conjunction* as a logical constant of the logic we stand on. Namely, $\mathbf{F}[\mathfrak{P}]$ and $\mathbf{F}[\mathfrak{p}]$ could be interpreted as $\mathbf{F}[\mathfrak{P}_1, \cdots, \mathfrak{P}_s]$ and $\mathbf{F}[\mathfrak{p}_1, \cdots, \mathfrak{p}_s]$, respectively, for appropriately defind formulas $\mathfrak{P}_1, \cdots, \mathfrak{P}_s$ which would work as $(\mathfrak{p}\to\mathfrak{q})\to\mathfrak{p}_i$ $(i=1, \cdots, s)$. This can be interpreted as

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1) Here we call any pair of formal systems *distinct* if and only if there is a proposition which is provable in one of the systems but unprovable in the other.

2) Under the *implication logic*, we understand the logic having *implication* and admitting the following inference rules: (1) $\mathfrak{B}$ is deducible from $\mathfrak{A}$ and $\mathfrak{A}\to\mathfrak{B}$, (2) $\mathfrak{A}\to\mathfrak{B}$ is deducible if $\mathfrak{B}$ is deducible from $\mathfrak{A}$. It is the sentential part **LOS** of the primitive logic **LO**. As for **LO**, see [1] Ono.

$$\mathfrak{Q}_1 \to (\mathfrak{Q}_2 \to (\cdots \to (\mathfrak{Q}_t \to \mathfrak{p}_i) \cdots)),$$

where $\mathfrak{Q}_j$ stands for $\mathfrak{p} \to \mathfrak{q}_j$ *i.e.*

$$\mathfrak{p}_1 \to (\mathfrak{p}_2 \to (\cdots \to (\mathfrak{p}_s \to \mathfrak{q}_j) \cdots)).$$

Thus we have

Theorem. *Two formal systems* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are very-close if and only if we can find out a formal system* $\mathbf{F}$ *and a series of propositions* $\mathfrak{p}_1, \cdots, \mathfrak{p}_s; \mathfrak{q}_1, \cdots, \mathfrak{q}_t$ *such that* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *lie between* $\mathbf{F}[\mathfrak{P}_1, \cdots, \mathfrak{P}_s]$ *and* $\mathbf{F}[\mathfrak{p}_1, \cdots, \mathfrak{p}_s]$, *where each* $\mathfrak{p}_i$ *is defined by*

$$\mathfrak{P}_i \equiv \mathfrak{Q}_1 \to (\mathfrak{Q}_2 \to (\cdots \to (\mathfrak{Q}_t \to \mathfrak{p}_i) \cdots)),$$

$$\mathfrak{Q}_j \equiv \mathfrak{p}_1 \to (\mathfrak{p}_2 \to (\cdots \to (\mathfrak{p}_s \to \mathfrak{q}_j) \cdots)).$$

Proof. Firstly, let us suppose that $\mathbf{M}_1$ and $\mathbf{M}_2$ are *very-close*, namely, that there exists a formal system $\mathbf{N}$ that satisfies $\mathfrak{C}(\mathbf{M}_1, \mathbf{M}_2, \mathbf{N})$. Then, we can find out a formal system $\mathbf{F}$ and a series of propositions $\mathfrak{a}_1, \cdots, \mathfrak{a}_l; \mathfrak{b}_1, \cdots, \mathfrak{b}_m; \mathfrak{q}_1, \cdots, \mathfrak{q}_t$ such that $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{N}$ are equivalent to $\mathbf{F}[\mathfrak{a}_1, \cdots, \mathfrak{a}_l]$, $\mathbf{F}[\mathfrak{b}_1, \cdots, \mathfrak{b}_m]$, and $\mathbf{F}[\mathfrak{q}_1, \cdots, \mathfrak{q}_t]$, respectively. Now, we denote the series of $\mathfrak{a}_1, \cdots, \mathfrak{a}_l; \mathfrak{b}_1, \cdots, \mathfrak{b}_m$ by $\mathfrak{p}_1, \cdots, \mathfrak{p}_l; \mathfrak{p}_{l+1}, \cdots, \mathfrak{p}_s$ $(s=l+m)$.

Evidently, $\mathbf{M}_1$ as well as $\mathbf{M}_2$ is weaker than $\mathbf{F}[\mathfrak{p}_1, \cdots, \mathfrak{p}_s]$. Hence, we have only to prove that each of them is stronger than $\mathbf{F}[\mathfrak{P}_1, \cdots, \mathfrak{P}_s]$. The condition $\mathfrak{C}(\mathbf{M}_1, \mathbf{M}_2, \mathbf{N})$ means that the proposition set $\{\mathfrak{a}_1, \cdots, \mathfrak{a}_l; \mathfrak{x}_1, \cdots, \mathfrak{x}_n\}$ is equivalent to the proposition set $\{\mathfrak{q}_1, \cdots, \mathfrak{q}_t\}$ if and only if the proposition set $\{\mathfrak{b}_1, \cdots, \mathfrak{b}_m; \mathfrak{x}_1, \cdots, \mathfrak{x}_n\}$ is equivalent to the proposition set $\{\mathfrak{q}_1, \cdots, \mathfrak{q}_t\}$ for any finite number of propositions $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$. We shall now show that each $\mathfrak{P}_i$ $(i=1, \cdots, s)$ is deducible from $\mathfrak{a}_1, \cdots, \mathfrak{a}_l$ in $\mathbf{F}$. For the case $i=1, \cdots, l$, this is clear. For the case $i=l+v$ $(v=1, \cdots, m)$, we would like to show $\mathfrak{p}_{l+v}$ *i.e.* $\mathfrak{b}_v$ by assuming $\mathfrak{a}_1, \cdots, \mathfrak{a}_l; \mathfrak{Q}_1, \cdots, \mathfrak{Q}_t$. Taking $\mathfrak{q}_1, \cdots, \mathfrak{q}_t$ for $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ of $\mathfrak{C}(\mathbf{M}_1, \mathbf{M}_2, \mathbf{N})$, we know that $\{\mathfrak{a}_1, \cdots, \mathfrak{a}_l; \mathfrak{q}_1, \cdots, \mathfrak{q}_t\}$ is equivalent to $\{\mathfrak{q}_1, \cdots, \mathfrak{q}_t\}$ if and only if $\{\mathfrak{b}_1, \cdots, \mathfrak{b}_m; \mathfrak{q}_1, \cdots, \mathfrak{q}_t\}$ is equivalent to $\{\mathfrak{q}_1, \cdots, \mathfrak{q}_t\}$. Since $\{\mathfrak{a}_1, \cdots, \mathfrak{a}_l; \mathfrak{q}_1, \cdots, \mathfrak{q}_t\}$ is equivalent to $\{\mathfrak{q}_1, \cdots, \mathfrak{q}_t\}$, $\{\mathfrak{b}_1, \cdots, \mathfrak{b}_m; \mathfrak{q}_1, \cdots, \mathfrak{q}_t\}$ is equivalent to $\{\mathfrak{q}_1, \cdots, \mathfrak{q}_t\}$. Hence, each $\mathfrak{b}_v$ is deducible from $\mathfrak{q}_1, \cdots, \mathfrak{q}_t$. Therefore, we have only to show that $\mathfrak{q}_1, \cdots, \mathfrak{q}_t$ hold. Again, taking $\mathfrak{Q}_1, \cdots, \mathfrak{Q}_t$ for $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$ of $\mathfrak{C}(\mathbf{M}_1, \mathbf{M}_2, \mathbf{N})$, we know that $\{\mathfrak{a}_1, \cdots, \mathfrak{a}_l; \mathfrak{Q}_1, \cdots, \mathfrak{Q}_t\}$ is equivalent to $\{\mathfrak{q}_1, \cdots, \mathfrak{q}_t\}$ if and only if $\{\mathfrak{b}_1, \cdots, \mathfrak{b}_m; \mathfrak{Q}_1, \cdots, \mathfrak{Q}_t\}$ is equivalent to $\{\mathfrak{q}_1, \cdots, \mathfrak{q}_t\}$. Since $\{\mathfrak{b}_1, \cdots, \mathfrak{b}_m; \mathfrak{Q}_1, \cdots, \mathfrak{Q}_t\}$ is equivalent to $\{\mathfrak{q}_1, \cdots, \mathfrak{q}_t\}$, $\{\mathfrak{a}_1, \cdots, \mathfrak{a}_l; \mathfrak{Q}_1, \cdots, \mathfrak{Q}_t\}$ is equivalent to $\{\mathfrak{q}_1, \cdots, \mathfrak{q}_t\}$. Hence, $\mathfrak{q}_1, \cdots, \mathfrak{q}_t$ hold by assumption. Thus, $\mathbf{M}_1$ is proved to be stronger than $\mathbf{F}[\mathfrak{P}_1, \cdots, \mathfrak{P}_s]$. In the same way, we can prove that $\mathbf{M}_2$ is also stronger than $\mathbf{F}[\mathfrak{P}_1, \cdots, \mathfrak{P}_s]$.

Conversely, let us assume that there exist a formal system $\mathbf{F}$ and a series of propositions $\mathfrak{p}_1, \cdots, \mathfrak{p}_s; \mathfrak{q}_1, \cdots, \mathfrak{q}_t$ such that $\mathbf{M}_1$ and $\mathbf{M}_2$ lie between $\mathbf{F}[\mathfrak{P}_1, \cdots, \mathfrak{P}_s]$ and $\mathbf{F}[\mathfrak{p}_1, \cdots, \mathfrak{p}_s]$. Let us further assume that $\mathbf{M}_1$ and $\mathbf{M}_2$ are equivalent to $\mathbf{F}[\mathfrak{a}_1, \cdots, \mathfrak{a}_l]$ and $\mathbf{F}[\mathfrak{b}_1, \cdots, \mathfrak{b}_m]$, respectively. These assumptions imply clearly

(1) $\mathfrak{a}_u$ is deducible from $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ in $\mathbf{F}$ $(u=1, \cdots, l)$,

(2) $\mathfrak{b}_v$ is deducible from $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ in $\mathbf{F}$ $(v=1, \cdots, m)$,

(3) $\mathfrak{P}_i$ is deducible from $\mathfrak{a}_1, \cdots, \mathfrak{a}_l$ in $\mathbf{F}$ $(i=1, \cdots, s)$,

(4) $\mathfrak{P}_i$ is deducible from $\mathfrak{b}_1, \cdots, \mathfrak{b}_m$ in $\mathbf{F}$ $(i=1, \cdots, s)$.

In order to show that $\mathbf{M}_1$ and $\mathbf{M}_2$ are *very-close*, we show that $\{\mathfrak{a}_1, \cdots, \mathfrak{a}_l; \mathfrak{x}_1, \cdots, \mathfrak{x}_n\}$ is equivalent to $\{\mathfrak{Q}_1, \cdots, \mathfrak{Q}_t\}$ if and only if $\{\mathfrak{b}_1, \cdots, \mathfrak{b}_m; \mathfrak{x}_1, \cdots, \mathfrak{x}_n\}$ is equivalent to $\{\mathfrak{Q}_1, \cdots, \mathfrak{Q}_t\}$ for any finite number of propositions $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$. If we succeed to show this, we have only to take $\mathbf{N}$ as $\mathbf{F}[\mathfrak{Q}_1, \cdots, \mathfrak{Q}_t]$. Now, we show that $\{\mathfrak{b}_1, \cdots, \mathfrak{b}_m; \mathfrak{x}_1, \cdots, \mathfrak{x}_n\}$ is equivalent to $\{\mathfrak{Q}_1, \cdots, \mathfrak{Q}_t\}$ by assuming that $\{\mathfrak{a}_1, \cdots, \mathfrak{a}_l; \mathfrak{x}_1, \cdots, \mathfrak{x}_n\}$ is equivalent to $\{\mathfrak{Q}_1, \cdots, \mathfrak{Q}_t\}$. Clearly, $\mathfrak{b}_1, \cdots, \mathfrak{b}_m; \mathfrak{x}_1, \cdots, \mathfrak{x}_n$ imply each $\mathfrak{Q}_j$ $(j=1, \cdots, t)$ in $\mathbf{F}$ by (1). Also $\mathfrak{Q}_1, \cdots, \mathfrak{Q}_t$ imply $\mathfrak{a}_1, \cdots, \mathfrak{a}_l; \mathfrak{x}_1, \cdots, \mathfrak{x}_n$ by assumption. Hence, we can prove each $\mathfrak{b}_v$ $(v=1, \cdots, m)$ in $\mathbf{F}$ by (2) and (3). In the same way, we can prove that $\{\mathfrak{a}_1, \cdots, \mathfrak{a}_l; \mathfrak{x}_1, \cdots, \mathfrak{x}_n\}$ is equivalent to $\{\mathfrak{Q}_1, \cdots, \mathfrak{Q}_t\}$ by assuming that $\{\mathfrak{b}_1, \cdots, \mathfrak{b}_m; \mathfrak{x}_1, \cdots, \mathfrak{x}_n\}$ is equivalent to $\{\mathfrak{Q}_1, \cdots, \mathfrak{Q}_t\}$. Thus, $\mathbf{M}_1$ and $\mathbf{M}_2$ are proved to be *very-close*.

Remark. An example pair of distinct *very-close* formal systems is given by $\mathbf{F}[(\mathfrak{p}\rightarrow\mathfrak{q})\rightarrow\mathfrak{p}]$ and $\mathbf{F}[\mathfrak{p}]$ for any formal system $\mathbf{F}$ which does not admit $((\mathfrak{p}\rightarrow\mathfrak{q})\rightarrow\mathfrak{p})\rightarrow\mathfrak{p}$. It is also remarkable that there is no pair of distinct *very-close* formal systems standing on any one of K-series logics.[3] Namely, if we assume Peirce's rule in $\mathbf{F}$, two systems $\mathbf{F}[\mathfrak{P}_1, \cdots, \mathfrak{P}_s]$ and $\mathbf{F}[\mathfrak{p}_1, \cdots, \mathfrak{p}_s]$ are proved to be mutually equivalent as follows: Namely, let us assume that Peirce's rule holds in $\mathbf{F}$. Then, we shall show by induction on t that each $\mathfrak{p}_i$ $(i=1, \cdots, s)$ is deducible from $\mathfrak{P}_i \equiv \mathfrak{Q}_1\rightarrow(\mathfrak{Q}_2\rightarrow(\cdots\rightarrow(\mathfrak{Q}_t\rightarrow\mathfrak{p}_i)\cdots))$ in $\mathbf{F}$. To show this, let us suppose $\mathfrak{P}_i$. $\mathfrak{Q}_2\rightarrow(\mathfrak{Q}_3\rightarrow(\cdots\rightarrow(\mathfrak{Q}_t\rightarrow\mathfrak{p}_i)\cdots))$ implies $\mathfrak{p}_i$ by assumption of induction. Hence, $\mathfrak{Q}_1\rightarrow\mathfrak{p}_i$ holds. We can easily see that $\mathfrak{Q}_1$ is equivalent to $\mathfrak{p}_i\rightarrow\mathfrak{Q}_1$, so $(\mathfrak{p}_i\rightarrow\mathfrak{Q}_1)\rightarrow\mathfrak{p}_i$ holds. Since we assume Peirce's rule in $\mathbf{F}$, $\mathfrak{p}_i$ is deducible in $\mathbf{F}$.

Reference

[1] Ono, K.: On universal character of the primitive logic. Nagoya Math. J., **27**(1), 331-353 (1966).

3) See [1] Ono.