39. *A Note on Jacobi Fields of δ-Pinched Riemannian Manifolds*

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In [1] M. Berger stated a theorem relating to Jacobi fields of a complete δ-pinched Riemannian manifold which is an extension of Rauch's metric comparison theorem [3]. This theorem is equivalent to the following

Proposition A. *Let M be a complete Riemannian manifold whose sectional curvature K satisfies the inequality*

(1) $$0<\delta\leqq K\leqq 1$$

and X be any Jacobi field along a geodesic $x=\gamma(s)$ parameterized with arc length s such that

(2) $$\langle X(0), \gamma'(0)\rangle=0,\ X'(0)=0, \qquad \|X(0)\|=1,$$

then

(3) $$\|X(s)\|\leqq\cos\sqrt{\delta}s \quad \textit{for} \quad 0\leqq s\leqq\pi/2.$$

Berger's proof of Theorem 1 in [1] is due to the principle of variation analogous to Rauch's method in [2] but it is not clear whether this theorem is true or not by his exposition only. Since it can be shown that Proposition A is true in the case dim $M=2$, it may be considered that it is a conjecture in the case dim $M>2$. In this note, the author will show that this proposition holds for a locally symmetric Riemannian manifold.

Let M be an n-dimensional Riemannian manifold and X a Jacobi field along a geodesic $x=\gamma(s)$ parameterized with arc length s. Then, X satisfies the following equation

(4) $$\frac{D^2X}{ds^2}+R\Big(\frac{d\gamma}{ds}, X\frac{d\gamma}{ds}\Big)=0,$$

where D denotes the covariant differentiation of M, $\frac{d\gamma}{ds}$ the tangent vector of the curve $x=\gamma(s)$ and R the curvature tensor field of M.

Now, let k be a positive constant and suppose that

$$X(s)\neq 0,\ y(s)\equiv\cos ks\neq 0$$

in an interval $0\leqq s<s_0$. In this interval, we put

(5) $$\varphi=\frac{\langle X, y'X-yX'\rangle}{\|X\|},$$

where $\langle,\rangle$ denotes the inner product in M and $X'=\frac{DX}{ds}$. If we have

$\langle X(0), \gamma'(0)\rangle = 0$, then $\langle X(s), \gamma'(s)\rangle = 0$ for any s. In this case, by virtue of (4) and

$$\left\langle X, R\left(\frac{d\gamma}{ds}, X\frac{d\gamma}{ds}\right)\right\rangle = K\left(X, \frac{d\gamma}{ds}\right)\| X\|^2,$$

where $K\left(X, \frac{d\gamma}{ds}\right)$ is the sectional curvature of the plane element spanned by X and $\frac{d\gamma}{ds}$, we have in the interval $0 < s < s_0$

$$\begin{aligned}\varphi' &= \frac{1}{\|X\|}\left\{\langle X', y'X - yX'\rangle + \langle X, y''X - yX''\rangle\right\} \\ &\quad - \frac{1}{\|X\|^3}\langle X, y'X - yX'\rangle\langle X, X'\rangle \\ &= \frac{y}{\|X\|}\left\{-\|X'\|^2 - k^2\|X\|^2 + \left\langle X, R\left(\frac{d\gamma}{ds}, X\frac{d\gamma}{ds}\right)\right\rangle\right\} \\ &\quad + \frac{1}{\|X\|^3}\langle X, X'\rangle^2 \\ &= y\|X\|\left\{K\left(X, \frac{d\gamma}{ds}\right) - k^2 - \left(\frac{\|X'\|^2}{\|X\|^2} - \frac{\langle X, X'\rangle^2}{\|X\|^4}\right)\right\}.\end{aligned}$$

Let us define a unit vector field along the geodesic by

$$\xi = \frac{X}{\|X\|},$$

then

$$\xi' = \frac{X'}{\|X\|} - \frac{\langle X, X'\rangle X}{\|X\|^3},$$

and so

$$\|\xi'\|^2 = \frac{\|X'\|^2}{\|X\|^2} - \frac{\langle X, X'\rangle^2}{\|X\|^4}.$$

The equation above can be written as

$$\varphi' = y\|X\|\left\{K\left(X, \frac{d\gamma}{ds}\right) - k^2 - \|\xi'\|^2\right\}. \tag{6}$$

Lemma 1. *Under the same hypothesis of Proposition A, we have*

$$\|X(s)\| \geqq \cos s \quad \text{for} \quad 0 \leqq s \leqq \pi/2.$$

Proof. In (6), we put $k=1$, then

$$\varphi' = -y\|X\|\left\{1 - K\left(X, \frac{d\gamma}{ds}\right) + \|\xi'\|^2\right\} \leqq 0$$

Since $\varphi(0)=0$, we get

$$\varphi(s) \leqq 0,$$

and so

$$\frac{\langle X, X'\rangle}{\|X\|^2} \geqq -\frac{\sin s}{\cos s} \quad \text{for} \quad 0 \leqq s \leqq s_0.$$

$\|X(0)\| = (\cos s)_{s=0} = 1$ and the inequality above imply

$$\|X(s)\| \geqq \cos s \quad \text{for} \quad 0 \leqq s \leqq s_0.$$

From these arguments, we can extend the interval $0 \leqq s \leqq s_0$ to the interval $0 \leqq s \leqq \pi/2$.

Lemma 2. *Under the same hypothesis of Proposition A, if we have $\xi' = 0$ for $0 \leqq s \leqq \pi/2$, then*

$$\|X(s)\| \leqq \cos \sqrt{\delta} s \quad \text{for } 0 \leqq s \leqq \pi/2.$$

Proof. In (6), we put $k = \sqrt{\delta}$, then

$$\varphi' = y\|X\|\left\{K\left(X, \frac{d\gamma}{ds}\right) - \delta\right\} \geqq 0$$

and so

$$\frac{\langle X, X'\rangle}{\|X\|^2} \leqq -\frac{\sqrt{\delta} \sin \sqrt{\delta} s}{\cos \sqrt{\delta} s} \quad \text{for} \quad 0 \leqq s \leqq \pi/2,$$

from which we get

$$\|X(s)\| \leqq \cos \sqrt{\delta} s \quad \text{for } 0 \leqq s \leqq \pi/2.$$

Theorem 1. *Let M be a complete Riemannian manifold whose sectional curvature K satisfies the inequality*

$$0 < \delta \leqq K \leqq 1$$

and $x = \gamma(s)$ be a geodesic parameterized with arc length s. If there exist $n-1$ Jacobi fields $\underset{(\alpha)}{X}, \alpha = 1, 2, \cdots, n-1$, along the geodesic such that

$$\|\underset{(\alpha)}{X}(0)\| = 1, \ \underset{(\alpha)}{X}'(0) = 0, \ \langle \underset{(\alpha)}{X}(0), \gamma'(0)\rangle = 0$$

and $\underset{(\alpha)}{\xi} = \underset{(\alpha)}{X}/\|\underset{(\alpha)}{X}\|$ are parallelly displaced along the geodesic and orthogonal each other to. Then, for any Jacobi field X satisfying the condition (2), we have

$$\|X(s)\| \leqq \cos \sqrt{\delta} s \quad \text{for} \quad 0 \leqq s \leqq \pi/2.$$

Proof. By virtue of the assumption and Lemma 2, we have

$$\|\underset{(\alpha)}{X}(s)\| \leqq \cos \sqrt{\delta} s \quad \text{for} \quad 0 \leqq s \leqq \pi/2.$$

Since we can represent X as

$$X(s) = \sum_{\alpha=1}^{n-1} a_\alpha \underset{(\alpha)}{X}(s), \ \sum_{\alpha=1}^{n-1} a_\alpha^2 = 1,$$

we have

$$\|X(s)\|^2 = \sum_{\alpha=1}^{n-1} a_\alpha^2 \|\underset{(\alpha)}{X}(s)\|^2 \leqq \cos^2 \sqrt{\delta} s \sum_{\alpha=1}^{n-1} a_\alpha^2 = \cos^2 \sqrt{\delta} s,$$

hence

$$\|X(s)\| \leqq \cos \sqrt{\delta} s \quad \text{for} \quad 0 \leqq s \leqq \pi/2.$$

Theorem 2. *Let M be a complete Riemannian manifold whose sectional curvature satisfies the inequality*

$$0 < \delta \leqq K \leqq 1,$$

$x=\gamma(s)$ a geodesic parameterized with arc length s and X a Jacobi field along the geodesic such that

$$\|X(0)\|=1,\ X'(0)=0,\ \langle X(0), \gamma'(0)\rangle=0.$$

If $\dim M=2$ or M is locally symmetric, then

$$\|X(s)\|\leqq\cos\sqrt{\delta}\, s \qquad \text{for} \quad 0\leqq s\leqq\pi/2.$$

Proof. In the case $\dim M=2$, $\xi=X/\|X\|$ is always parallelly displaced along the geodesic for $0\leqq s\leqq\pi/2$ by Lemma 1 and $\gamma''(s)=0$. Hence, by Lemma 2, we have $\|X(s)\|\leqq\cos\sqrt{\delta}\, s$ for $0\leqq s\leqq\pi/2$.

In the case M is locally symmetric, we take an orthogonal frame $(\gamma(0), e_1(0), \cdots, e_{n-1}(0), e_n(0))$, $e_n(0)=\gamma'(0)$ at $x=\gamma(0)$ and parallelly displace this frame along the geodesic which we denotes $(\gamma(s), e_1(s), \cdots, e_{n-1}(s), e_n(s))$. If we put $X=\sum_i X_i(s)e_i(s)$, then the Jacobi equation (4) along the geodesic can be written as

$$(4') \qquad \frac{d^2X_i}{ds^2}+\sum_j R_{nijn}X_j=0,$$

where R_{ijhk} are the components of the curvature tensor with respect to the frame $(\gamma(s), e_1(s), \cdots, e_n(s))$. Since M is locally symmetric, R_{ijhk} are all constants along the geodesic. If we firstly choose $(\gamma(0), e_1(0), \cdots, e_n(0))$ so that

$$R_{nijn}=0 \qquad (i\neq j),$$

then we have $n-1$ Jacobi fields as follows

$$X(s)=(\cos\sqrt{R_{n\alpha\alpha n}}\, s)e_\alpha(s),\ \alpha=1, 2, \cdots, n-1,$$

which satisfy the condition in Theorem 1. Thus, we have

$$\|X(s)\|\leqq\cos\sqrt{\delta}\, s \qquad \text{for} \quad 0\leqq s\leqq\pi/2$$

by Theorem 1.

Remark. The author does not know whether Proposition A is true or not. If it is not true, we have to make a counter example in a suitable Riemannian manifold which is of dimension >2 and not locally symmetric. And we have to take a geodesic in the manifold along which we can not choose $n-1$ Jacobi fields $\underset{(\alpha)}{X}$, $\alpha=1, 2, \cdots, n-1$, as in Theorem 1.

References

[1] M. Berger: An extension of Rauch's metric comparison theorem and some applications. Illinois Math. J., **61**, 700–712 (1962).

[2] H. E. Rauch: A contribution to differential geometry in the large. Ann. of Math., **54**, 38–55 (1951).

[3] ——: Geodesics and curvature in differential geometry in the large. New York, Yeshiva University (1959).