69. *The Normality of the Product of Two Linearly Ordered Spaces*

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Introduction. Let X and Y be normal topological spaces. The product space $X\times Y$ is not necessarily normal. The problem of deciding when $X\times Y$ is normal, is interesting, in view of the fact that a Hausdorff topological space is normal if and only if any continuous real-valued function defined on any closed subset can be extended to a continuous function on the whole space. In this paper we shall settle this problem in the case where each factor space is a locally compact linearly ordered space. Our result extends the Ball's theorem [1], which assumes that one factor space is compact.

1. By a linearly ordered space we mean a linearly ordered set with the interval topology. It is well known that every such space is normal.

Let L be a non-empty linearly ordered space. An interior gap of L is a Dedekind cut $(A\mid B)$ of L such that $A\neq\phi, B\neq\phi$. A has no last point and B has no first point. If L has no first (last) point, there exists a left (right) end gap $(\phi\mid L)$ $((L\mid\phi))$. We denote by L' the set of all gaps of L and by $\bar{L}$ the sum of L and L'. $\bar{L}$ is a compact linearly ordered space. To denote intervals of L, we shall employ the Bourbaki's symbols, $[\ ,\ ]$, $]\leftarrow, [$, etc. Boundaries of an interval of L may be gaps of L as well as points of L.

We define $\rho(L)$ as follows. In case L has a right end gap which is not a limit of interior gaps of L, we put $\rho(L)=\alpha$, where α is a regular initial ordinal such that there exists an increasing sequence $\{x_\lambda; \lambda<\alpha\}$ of points of L which is cofinal with L. In all other cases we put $\rho(L)=0$, more precisely, $\rho(L)=0$ in the following three cases; (1) $L=\phi$, (2) L has a last point, (3) L has a right end gap which is a limit of interior gaps of L.

Let u be any point or gap of L. We put $\rho_-(u)=\rho(\ ]\leftarrow, u[\ )$ and $\rho_+(u)=\rho(\ ]u,\rightarrow[\ ^*)$, where $*$ signifies the inversely ordered set. Finally we define $\tau(L)$ for locally compact linearly ordered space L, as follows. $\tau(L)=$the smallest regular initial ordinal α such that $\rho_-(x)<\alpha$ and $\rho_+(x)<\alpha$ for every point $x\in L$.

We shall say that a point or gap u of L is of type α, if either $\rho_-(u)=\alpha$ or $\rho_+(u)=\alpha$. We denote by ω the first infinite ordinal.

Lemma 1. *Let X and Y be linearly ordered spaces. If X has a point of type α, and Y has a gap of type α, for some regular initial ordinal $\alpha>\omega$, then $X\times Y$ is not normal.*

Proof. Let us suppose $x\in X$, $v\in Y'$, and $\rho_-(x)=\alpha$, $\rho_-(v)=\alpha$. Let $W(\alpha)$ be the linearly ordered space of all ordinals less than α. We can find strictly increasing and continuous mappings $f: W(\alpha)\to X$ and $g: W(\alpha)\to Y$, with $\lim_{\lambda<\alpha} f(\lambda)=x$, $\lim_{\lambda<\alpha} g(\lambda)=v$. Consider the following two subsets of $X\times Y$; $\{(f(\lambda), g(\lambda)); \lambda<\alpha\}$ and $\{x\}\times Y$. These are disjoint closed subsets, but cannot be separated by disjoint open sets, since $\alpha>\omega$. Hence $X\times Y$ is not normal.

Lemma 2. *Let X be a linearly ordered space without any gaps other than the right end gap u, and let Y be a compact linearly ordered space. If $\rho_-(u)\geqq\tau(Y)$, then $X\times Y$ is normal.*

For a proof of this lemma, see [1].

2. This section is devoted to a proof of the following lemma.

Lemma 3. *Let X and Y be linearly ordered spaces without any gaps other than the right end gaps u and v respectively. If $\rho_-(u)=\alpha$, $\rho_-(v)=\alpha$, and $\alpha\geqq\tau(X)$, $\alpha\geqq\tau(Y)$ for some $\alpha>\omega$, then $X\times Y$ is normal.*

Proof. (i). We can find a strictly increasing and continuous mapping $f: W(\alpha)\to X$, such that the image $f(W(\alpha))$ is cofinal with X, and $f(0)=$ the first point of X. Using f, we can construct an increasing and left continuous mapping $\hat{f}: X\to W(\alpha)$ as follows. $\hat{f}(x)=\lambda$, if $f(\lambda)=x$, and $\hat{f}(x)=\lambda+1$, if $f(\lambda)<x<f(\lambda+1)$. Then clearly $\hat{f}(f(\lambda))=\lambda$ and $f(\hat{f}(x))\geqq x$. Similarly, we can define $g: W(\alpha)\to Y$ and $\hat{g}: Y\to W(\alpha)$, with the same properties as f and $\hat{f}$. Let us consider the composed mappings $\varphi=g\circ\hat{f}$ and $\psi=f\circ\hat{g}$. Then obviously φ and ψ are increasing and left continuous, and we have

(2, 1) $\psi\circ\varphi(x)\geqq x$ for any $x\in X$ and

$\phi\circ\psi(y)\geqq y$ for any $y\in Y$.

We define subsets Z_1, Z_2, D of $X\times Y$ as follows.

$$Z_1=\{(x, y)\,;\, y\geqq\varphi(x)\},\ Z_2=\{(x, y)\,;\, x\geqq\psi(y)\},$$

$D=\bigcup_{\lambda<\alpha} Q(\lambda)$, where $Q(\lambda)$ denotes a point $(f(\lambda), g(\lambda))$ if λ is a limit ordinal or 0, and $Q(\lambda)=]f(\mu), f(\lambda)]\times]g(\mu), g(\lambda)]$ if λ is an isolated ordinal succeeding μ. Then we can see

(2,2) $X\times Y=Z_1\cup Z_2\cup D$.

(ii). Let us agree in this section that for any subset A of $X\times Y$, $\bar{A}'$ denotes the closure of A in the enlarged space $\bar{X}\times\bar{Y}$.

We now wish to prove that, for any closed subset $F\subset X\times Y$,

(2,3) $\bar{F}'\ni(u, v)$ implies $\overline{F\cap D}'\ni(u, v)$,

where u and v are the right end gaps of X and Y as before.

Suppose $\overline{F\cap D}'\not\ni(u, v)$. Then there is a neighborhood

$$N=\{(x, y)\,;\, x_0<x,\, y_0<y\}$$

of the point (u, v) for which $F_\cap D_\cap N=\phi$. On the other hand, from (2,2), $\bar{F}'=\overline{F_\cap Z_1'}\cup\overline{F_\cap Z_2'}\cup\overline{F_\cap D'}$. Since $\bar{F}'\ni(u,v)$, either $\overline{F_\cap Z_1'}\ni(u,v)$ or $\overline{F_\cap Z_2'}\ni(u, v)$.

We may assume $\overline{F_\cap Z_1'}\ni(u, v)$. Then we can find a sequence of points $\{(x_n, y_n)\,;\, n=1, 2, \cdots<\omega\}\subset F_\cap Z_{1\cap}N$ such that $x_n>\psi(y_{n-1})$, for every n. Then it follows from the definition of Z_1 and from (2, 1) that $y_n\geqq\varphi(x_n)\geqq\varphi\circ\psi(y_{n-1})\geqq y_{n-1}$ and $x_n>\psi(y_{n-1})\geqq\psi\circ\varphi(x_{n-1})\geqq x_{n-1}$. Putting $\bar{x}=\lim x_n$, $\bar{y}=\lim y_n$, we can see easily that $\bar{x}=f(\beta)$, $\bar{y}=g(\beta)$ where $\beta=\hat{f}(\bar{x})=\hat{g}(\bar{y})$. Hence $(\bar{x}, \bar{y})\in F_\cap D_\cap N$. This contradicts the choice of N. Thus (2,3) is proved.

(iii). We shall prove next that

(2,4) if F_1 and F_2 are closed subsets of $X\times Y$ and $F_{1\cap}F_2=\phi$, then $\bar{F}_1'{}_\cap\bar{F}_2'\not\ni(u, v)$.

Suppose $\bar{F}_{1\cap}'\bar{F}_2'\ni(u, v)$. Then by virtue of (2,3), we have

$$\overline{F_{1\cap}D'}_\cap\overline{F_{2\cap}D'}\ni(u, v).$$

Hence we can choose a sequence $\lambda_0<\lambda_1<\cdots<\lambda_n<\cdots$, such that $Q(\lambda_{2n})_\cap F_1\neq\phi$ and $Q(\lambda_{2n+1})_\cap F_2\neq\phi$ for every n, by the definition of D. Then we can see that $Q(\lim\lambda_n)$ is a point contained in $F_{1\cap}F_2$. This contradicts the hypothesis $F_{1\cap}F_2=\phi$, and (2,4) is proved.

(iv). Finally we shall prove that for any disjoint closed subsets F_1 and F_2 of $X\times Y$, there exist open sets G_1 and G_2 such that $G_1\supset F_1$, $G_2\supset F_2$, $G_{1\cap}G_2=\phi$.

By (2,4), at least one of the sets $\bar{F}_1'$, $\bar{F}_2'$ does not contain (u, v). Let us suppose $\bar{F}_2'\not\ni(u, v)$. Then there exists a neighborhood

$$N=\{(x, y)\,;\, x_0\leqq x,\, y_0\leqq y\}$$

of (u, v) such that $F_{2\cap}N=\phi$. Lemma 2 shows that subsets $]\leftarrow,x_0]\times Y$ and $[x_0,\rightarrow[\times]\leftarrow,y_0]$ of $X\times Y$ are normal. Hence it follows that their sum, denoted by K, is normal. Since $F_{1\cap}K$ and $F_{2\cap}K=F_2$ are disjoint closed subsets of K, there exist open subsets U_1, U_2 of $X\times Y$, such that $U_1\supset F_{1\cap}K$, $U_2\supset F_2$, $U_{1\cap}U_{2\cap}K=\phi$. Then

$G_1=U_1\cup(\,]x_0,\rightarrow[\times]y_0,\rightarrow[\,)$, $G_2=U_{2\cap}$ (complement of N in $X\times Y$) are desired open sets. Thus the lemma 3 is verified.

3. Definition. *A locally compact linearly ordered space X is said to be of quasi-countable type, if for every gap u of X, $\rho_-(u)\leqq\omega$ and $\rho_+(u)\leqq\omega$.*

It is easily seen that a locally compact linearly ordered space is paracompact if and only if it is of quasi-countable type.

Definition. *A locally compact linearly ordered space X is said to be regular of type α, (where α is a regular initial ordinal $>\omega$), if the following three conditions are satisfied.*

(3,1) *For any gap u of X, $\rho_-(u)>\omega$ implies $\rho_-(u)=\alpha$ and $\rho_+(u)>\omega$*

implies $\rho_+(u)=\alpha$.

(3,2) *There exists a gap* u_0 *of* X *such that* $\rho_-(u_0)=\alpha$ *or* $\rho_+(u_0)=\alpha$.

(3,3) $\tau(X)\leqq\alpha$.

Theorem. *Let* X *and* Y *be locally compact linearly ordered spaces. Then* $X\times Y$ *is normal, if and only if one of the following two conditions is satisfied.*

(a) *At least one of* X *and* Y, *say* Y, *is of quasi-countable type and for any gap* u *of* X, $\rho_-(u)>\omega$ *implies* $\rho_-(u)\geqq\tau(Y)$ *and* $\rho_+(u)>\omega$ *implies* $\rho_+(u)\geqq\tau(Y)$.

(b) *There exists an uncountable regular initial ordinal* α, *such that both* X *and* Y *are regular of type* α.

Proof of the necessity. We suppose that $X\times Y$ is normal.

(a′) In case Y is of quasi-countable type. If $\omega<\rho_-(u_0)<\tau(Y)$ for some gap u_0 of X, then for some point $\bar{y}$ of Y, $\omega<\rho_-(u_0)\leqq\rho_-(\bar{y})$ or $\rho_+(\bar{y})$. It is easy to see that there exists a point y_0 in a neighborhood of $\bar{y}$, such that $\rho_-(u_0)=\rho_-(y_0)$ or $\rho_+(y_0)$. Then Lemma 1 shows that $X\times Y$ is not normal, contradicting our assumtion. Therefore if $\omega<\rho_-(u)$ for some gap u of X, then $\tau(Y)\leqq\rho_-(u)$, and similarly for ρ_+.

(b′) In case neither X nor Y is of quasi-countable type. There are gaps $u_0\in X'$, $v_0\in Y'$ such that $\max\{\rho_-(u_0), \rho_+(u_0)\}>\omega$ and $\max\{\rho_-(v_0), \rho_+(v_0)\}>\omega$. We may assume $\rho_-(u_0)>\omega$ and $\rho_-(v_0)>\omega$. At first we must show that $\rho_-(u_0)=\rho_-(v_0)$. Suppose, for instance, $\rho_-(u_0)>\rho_-(v_0)$. Then by definition of ρ_-, there exists an interval $[x_0, u_0[$, which has no interior gaps, and we can find a point x_1 in this interval such that $\rho_-(x_1)=\rho_-(v_0)$. Hence by Lemma 1 we are lead to a contradiction. Thus we can conclude $\rho_-(u_0)=\rho_-(v_0)$. We denote this value by α. The above observation shows also that for any $u\in X'$ and $v\in Y'$, $\rho_\pm(u)$, $\rho_\pm(v)$ must take the same value α, as far as they are uncountable. Next, if $\rho_-(u_0)=\alpha<\tau(Y)$, then $X\times Y$ cannot be normal as is seen in (a), hence we have $\alpha\geqq\tau(Y)$, and similarly $\alpha\geqq\tau(X)$. Thus X and Y are regular of type α.

Proof of the sufficiency. Let $X=\cup I$ be the decomposition of X as a sum of disjoint intervals, where each interval I has no interior gaps and I is maximal with respect to this property. It follows easily that each I is both open and closed, since X is locally compact. Let $Y=\cup J$ be the analogous decomposition of Y. Then each product $I\times J$ is also an open and closed subset of $X\times Y$. Therefore it is sufficient to prove that each $I\times J$ normal. Now we take a point (a, b) in $I\times J$ and put $I_1=[a,\rightarrow[$, $I_2=]\leftarrow,a]^*$, $J_1=[b,\rightarrow[$, $J_2=]\leftarrow, b]^*$. Again if each $I_i\times J_j$, $i, j=1, 2$, is normal, $I\times J$ is also normal. We shall prove the normality of $I_i\times J_j$ in each case (a) and (b), writing anew $I\times J$ instead of $I_i\times J_j$.

Case (a). If J has a right end gap v, then $\rho_-(v)=\omega$ by assumption. Hence we can take a sequence $y_0<y_1<\cdots<y_n<\cdots$ with limit v, and $y_0=$ the first point of J. If each $I\times[y_{n-1}, y_n]$ is normal, it is evident that $I\times J$ is normal. Hence this case is reduced to the case where J is compact. Similarly if the interval I has a right end gap of type ω, we can replace I by a compact interval. Therefore it suffices to prove for only two cases, (1) I and J are compact, (2) I has a right end gap u with $\rho_-(u)\geqq\tau(Y)$ and J is compact. Since $\tau(J)\leqq\tau(Y)$, normality of $I\times J$ follows from lemma 2.

Case (b). If we apply the reduction mentioned in case (a), we can see that normality for this case follows from Lemma 3. Hence our theorem is proved.

References

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