67. *Relations between Volumes and Measures*

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Introduction. A function v defined on a family V of sets of a space X is called a *volume* if the following two conditions are satisfied:

(1) The family V is a prering, that is the family is non-empty and if $A, B \in V$ then $A \cap B \in V$ and

$$A \backslash B = C_1 \cup \cdots \cup C_k,$$

where $C_j \in V$ are disjoint sets.

(2) The function v is non-negative, finite-valued, and countably additive on the prering V.

A volume v is called *upper complete* if the condition $A_n \in V$ and $\sum_n v(A_n) < \infty$ implies $A = \bigcup_n A_n \in V$. If in addition the condition $A \subset B \in V$ and $v(B) = 0$ implies $A \in V$, then the volume v is called *complete*.

In §1 we investigate upper complete volumes. The main result of the section is that upper complete volumes are in 1-1 correspondence with δ-finite measures. In this section we also establish the existence of *minimal extensions* of upper complete volumes to measures.

In §2 we prove that for every volume v there exists the *smallest complete measure* being an extension of the volume v. This result permits us to prove the classical theorem on extension of volumes. Namely if v is a volume on a prering V and M is the smallest σ-ring containing V, then there exists one and only one measure μ on M being an extension of the volume v. It is established that the completion of the measure μ_c yields the *smallest complete measure* being an extension of the volume v.

It is also established that for every volume v there exists the *smallest upper complete volume* being an extension of the volume v. The existence of the smallest complete volume satisfying this condition was established in [9].

§1. Relations between upper complete volumes and measures.

Theorem 1. *Let v be an upper complete volume on V and let M_0 be the family of all sets of the form $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in V$. Then the family M_0 is a sigma-ring.*

Theorem 2. *Let v be an upper complete volume on V and let*

M_0 be the sigma-ring

$$M_0=\{A : A=\bigcup_{n=1}^{\infty} A_n,\ A_n \in V\}.$$

There exists one and only one measure μ_0 on M_0 being an extension of the volume v. The measure is given by the formula

$$\mu_0(A)=\sup\{v(B) : B\subset A,\ B\in V\} \text{ for all } A\in M_0.$$

Denote by p the operator mapping an upper complete volume v into the measure μ_0 defined in Theorem 2.

If μ, η are two functions then the order relation $\eta\subset\mu$ will mean that the function μ is an extension of the function η.

If F is a family of functions we say that a function μ_0 is the smallest in the family F whenever

$$\mu_0\in F \text{ and } \mu_0\subset\mu \text{ for all } \mu\in F.$$

Theorem 3. *Let v be an upper complete volume on V and let $\mu_0=pv$. Then μ_0 is the smallest measure being an extension of the volume v.*

Theorem 4. *Let v be an upper complete volume on V and let $\mu_0=pv$. Then μ_0 is the smallest measure μ such that its finite part is the volume v, that is such that $t\mu=v$.*

Let i be the operator mapping a complete measure μ into the complete integral seminorm $J=i\mu$ defined as in [12].

Theorem 5. *Let J be a complete integral seminorm and $v=gJ$ the corresponding complete volume. Then $\mu_0=pv$ is the smallest complete measure generating J, that is the smallest complete measure μ such that $J=i\mu$.*

We say that a measure μ on a sigma-ring M is *sigma-finite* if for every set $A\in M$ there exists a sequence of sets $A_n\in M$ such that

$$A=\bigcup_n A_n \text{ and } \mu(A_n)<\infty.$$

Theorem 6. *Let v be an upper complete volume. Then $\mu=pv$ if and only if μ is a sigma-finite measure such that $v=t\mu$.*

We have noticed that a measure μ is complete if and only if the volume $v=t\mu$ is complete. From the proven theorems we see that the relations $\mu=pv$ and $J=i\mu$ establish 1-1 correspondence between the following: *any complete volume v, any complete sigma-finite measure μ, and any complete integral seminorm.*

§ 2. Extensions of volumes to measures and relations between the integral seminorms generated by them.

If v is a volume then its completion is defined by $v_c=g(i(v))$, that is

$$v_c(A)=\int\chi_A\,dv \text{ for } A\in V_c$$

where

$$V_c = \{A \subset X : \chi_A \in L(v, R)\}.$$

The volume v_c can be characterized as the *smallest complete volume* being an extension of the volume v according to Theorem 1, § 1, [8] and Theorem 5, § 2, [9].

Theorem 1. *Let v be a volume and $\mu_0 = pv_c$. Then μ_0 is the smallest complete measure being an extension of the volume v.*

Let μ be a measure on a sigma-ring M. Denote by N_μ the family of null sets generated by this measure. A set A belongs to this family if and only if there exists a set $B \in M$ such that $A \subset B$ and $\mu(B)=0$.

Denote by M_c the family of all sets $A = B \div C$, where $B \in M$ and $C \in N_\mu$, and put $\mu_c(A) = \mu(B)$. We remind the reader that the symmetric difference operation is defined by the formula $B \div C = (B \backslash C) \cup (C \backslash B)$. Any *ring of sets* with the *symmetric difference operation* forms a *group*.

The function μ_c is a measure called the Lebesgue extension of the measure μ. (See [14], [15]). This measure will be called the completion of the measure μ.

Theorem 2. *If μ is a measure on a sigma-ring M then the family M_c is a sigma-ring and the completion μ_c of μ considered on M_c is the smallest complete measure being an extension of the measure μ.*

Theorem 3. *Let μ be a measure and v its finite part, that is the function v represents the restriction of the measure μ to the family $V = \{A \in M : \mu(A) < \infty\}$, mhere M denotes the sigma-ring being the domain of the measure μ. Then the finite part of the measure μ_c coincides mith the completion v_c of the volume v.*

Denote by t the operator mapping a measure μ into its finite part $v = t\mu$.

Theorem 4. *Let v be a volume on a prering V and $\mu_0 = pv_c$. Let M_0 be the sigma-ring being the domain of the measure μ_0. If M is a sigma-ring such that $V \subset M \subset M_0$, then there exists unique measure μ being an extension of the volume v from the prering V onto the sigma-ring M.*

The measure is given by the formula

$$\mu(A) = \mu_0(A) \text{ for all } A \in M.$$

Moreover we have $\mu_c = \mu_0$.

Let E be a family of sets of a space X. Assume that F is the family of all σ-rings M containing E.

Notice that the intersection

$$M_1 = \cap M (M \in F)$$

is a sigma-ring. Since

$$M_1 \subset M \text{ for all } M \in F$$

therefore M_1 is the *smallest sigma-ring containing* E. We shall say that M_1 is generated by E and we shall write $M_1=\sigma$-ring (E).

Let V be a prering and v be a volume on it. Put $M=\sigma$-ring (V). We see that $M\subset M_0$, where M_0 is the domain of the measure $\mu_0=pv_c$. According to Theorem 4 there exists a unique measure μ on M such that $v\subset\mu$.

Theorem 5. *Let v be a volume on a prering V, $M=\sigma$-ring (V), and μ the measure on M being an extension of the volume v. Then μ is the smallest measure being an extension of the volume v. We also have $\mu_c=\mu_0$ where $\mu_0=pv_c$.*

Theorem 6. *Let v be a volume and μ the smallest measure being an extension of the volume v. Then $w=t\mu$ is the smallest upper complete volume being an extension of the volume v.*

Let v be a volume on a prering V. Denote by S the family of all sets of the from $A=\bigcup_n A_n$ where A_n is a finite family of sets from the prering.

Denote by S_δ the family of all sets of the form $A=\bigcap_n A_n$ corresponding to all sequences $A_n\in S$.

Let $S_{\delta\sigma}$ be the family of all sets of the form $A=\bigcup_n A_n$ corresponding to all sequences $A_n\in S_\delta$.

Let N_v denote as usual the family of null sets generated by the volume v (see [1]).

Denote by M_v the family of all sets of the form $A=B\div C$, where $B\in S_{\delta\sigma}$ and $C\in N_v$.

In [5], § 3 we have proven that the family M_v is a sigma-ring and that the volume v has a unique extension to a measure μ_v on M_v. This measure μ_v is complete according to Theorem 4-(5), § 3, [5].

Theorem 7. *Let v be a volume. Then the measure μ_v being an extension of the volume v from the prering V onto the sigma-ring M_v is the smallest complete measure η such that $v\subset\eta$.*

Theorem 8. *Let v be a volume and μ the smallest measure being an extension of the volume v. Then the integral seminorm generated by the volume v coincides with the integral seminorm generated by the completion μ_c of the measure μ, that is $J=iv=i\mu_c$.*

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