# 63. *B(C)-Spaces and B-Completeness**)

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Let $C$ be a class of real or complex locally convex Hausdorff topological vector (abbreviated to l.c.) spaces. An l.c. space $E$ is said to be a $B(C)$-space (resp. $B_r(C)$-space) if, for each $F$ in $C$, a linear continuous and almost open (resp. one-to-one) mapping of $E$ into $F$ is open (cf: [1], p. 83). If $C=\mathcal{A}$, the class of *all* l.c. spaces, then $B(\mathcal{A})$-and $B_r(\mathcal{A})$-spaces are respectively called $B$-complete and $B_r$-complete spaces (cf: [1], Chap. 7). Similarly, by specializing $C$, one defines $B(\mathcal{T})$-and $B(Q)$-spaces, where $\mathcal{T}$ and $Q$ denote the classes of all barrelled and quasibarrelled l.c. spaces, respectively.

It is known ([1], Chapter 7, Theorem 5) that a barrelled $B(\mathcal{T})$-space (resp. $B_r(\mathcal{T})$-space) is $B$-complete (resp. $B_r$-complete). In this note, we observe that under suitable conditions an l.c. space $E$ in $C$ which is also a $B(C)$-space (resp. $B_r(C)$-space) is $B$-complete (resp. $B_r$-complete). This extends the preceding result. From this, among other results, we derive a necessary and sufficient condition for a metrizable (resp. normed) l.c. space to be a Fréchet (resp. Banach) space, by using the open mapping theorem. At the end, we note that these ideas can be used more generally for topological spaces. As an instance, we show that the image in a Hausdorff space of a locally compact space under a continuous and almost open mapping is locally compact, thus generalizing a well-known fact.

We shall use the notations and definitions of [1].

Let $C$ be a class of l.c. spaces. It is understood by the definition of l.c. spaces, that each l.c. space in $C$ is *Hausdorff*. We say that the class $C$ satisfies the "invariant" property (I) if the following holds:

*If for any l.c. space $F$ there is a linear continuous almost open mapping of a member $E\in C$ into $F$, then it follows that $F$ is also in $C$.*

It is known that the classes $\mathcal{T}$ and $Q$ of barrelled and quasibarrelled l.c. spaces satisfy (I) (cf: [1], p. 20, Prop. 5 and p. 22, (5)(d)). The classes of $B$-complete l.c. spaces as well as that of $S$-spaces (cf: [1], Chap. 6) also satisfy (I) (cf: [1], p. 47, and p. 77). We show the following:

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**Proposition 1.** *The class $\mathcal{M}$ of all metrizable l.c. spaces satisfies* (I). *In other words, if there is a linear continuous almost open mapping of a metrizable l.c. space $E$ into any l.c. space $F_v$ then $F_v$ is also metrizable.*

**Proof.** Let $E_u$ be a metrizable l.c. space, $F_v$ any Hausdorff l.c. space, and $f$ a linear continuous and almost open mapping of $E$ into $F_v$. Let $\{U_n\}$ be a countable fundamental system of circled convex closed neighborhoods of 0 in $E_u$ such that $U_{n+1}+U_{n+1}\subset U_n$ for each $n\geq 1$ and $\bigcap_{n=1}^{\infty} U_n=\{0\}$.

For each $U_n$, put $\overline{f(U_n)}=W_n$. Then each $W_n$ is a neighborhood of 0 in $F_v$ because $f$ is almost open. Clearly each $W_n$ is convex, circled, $v$-closed and absorbing. Moreover, $W_{n+1}+W_{n+1}\subset W_n$. Let $w$ be the locally convex topology which has the sequence $\{W_n\}$ as a fundamental system of convex neighborhoods of 0 in $F$. It is clear that the initial topology $v$ is finer than $w$ on $F$.

Moreover, for each convex circled $v$-closed $v$-neighborhood $V$ of 0 in $F$, there exists a $U_n$ such that $f(U_n)\subset V$, because $f$ is continuous. Hence taking $v$-closures in the last inclusion relation, we see that $W_n\subset V$. This shows that $w$ is finer than $v$ and hence $v=w$ on $F$, because $v$ is already finer than $w$. Since $v$ is Hausdorff, so is $w$ and thus $F_v=F_w$ is a metrizable l.c. space because $w$ is metrizable on account of a sequence $\{W_n\}$ of convex neighborhoods as its base. Thus $F_v$ is in $\mathcal{M}$.

A consequence of the above proposition is the following:

**Corollary 1.** *The class $\mathcal{N}$ of all normed spaces satisfies property* (I).

**Proposition 2.** *Let $C$ be a class of l.c. spaces satisfying* (I). *Then every member $E$ in $C$ which is also a $B(C)$-space* (*resp. $B_r(C)$-space*) *is $B$-complete* (*resp. $B_r$-complete*).

**Proof.** Let $F$ be any l.c. space and let $f$ be a linear continuous and almost open mapping of $E$ into $F$. Since $C$ satisfies (I), $F$ is also in $C$. But $E$ being a $B(C)$-space, it follows that $f$ is open and hence $E$ is $B$-complete.

Similar arguments work for $B_r$-completeness by assuming $f$ to be one-to-one.

The following corollaries are immediate from proposition 2. In corollaries 3 and 4 we use the fact that a $B$-complete l.c. space is complete ([1], p. 64, Prop. 10).

**Corollary 2.** *Every quasibarrelled $B(Q)$-space* (*resp. $B_r(Q)$-space*) *is $B$-complete* (*resp. $B_r$-complete*).

**Corollary 3.** *Every metrizable $B(\mathcal{M})$-space or $B_r(\mathcal{M})$-space is a Fréchet space.*

**Corollary 4.** *Every normed $B(\mathfrak{N})$-space or $B_r(\mathfrak{N})$-space is a Banach space.*

By using the open mapping theorem, we can obtain a criterion for a metrizable or normed space to be complete as in the following:

**Proposition 3.** (a) *A metrizable l.c. space $E$ is a Fréchet space (in particular complete) if and only if $E$ is a $B(\mathcal{M})$-space.*

(b) *A normed space is a Banach space if and only if it is a $B(\mathfrak{N})$-space.*

**Proof.** (a) The "if" part follows from Corollary 3. For the "only if" part, let $E$ be a Fréchet space. Let $F$ be any metrizable l.c. space and $f$ a linear continuous almost open mapping of $E$ into $F$. Then by the open mapping theorem ([1], p. 36, Theorem 2) $f$ is open. Hence $E$ is a $B(\mathcal{M})$-space.

(b) Similar arguments as those for (a) work.

**Remark.** In Corollaries 3 and 4, it is clear that one can replace $\mathcal{M}$ and $\mathfrak{N}$ by $Q$, since each metrizable or normed l.c. space is quasi-barrelled and since each $B(Q)$-space is a $B(\mathcal{M})$-and hence $B(\mathfrak{N})$-space. However, in the same corollaries, $\mathcal{M}$ and $\mathfrak{N}$ may not be replaced by $\mathfrak{T}$, as shown in the following examples.

**Example 1.** Let $E=L^1(N)$, the space of all real sequences $x=\{x_n\}$ such that $\sum_{n=1}^{\infty}|x_n|<\infty$. $E$, as a subset of $R^N$, the space of all real sequences, is a metrizable *dense* subspace of $R^N$ in the metrizable l.c. topology induced from $R^N$. Hence $E$ is not a Fréchet space. However, it is known that $E$ is a $B(\mathfrak{T})$-space, see for instance ([1], Chapter 7, §6, Example 3).

**Example 2.** Let $F=R^{(N)}$, the space of all finite real sequences i.e., $x=\{x_n\}\in F$ if $x_n=0$ for all $n$ except for a finite number of $n$'s. $F$, with the norm: $\|x\|=\sup|x_n|$, is not a Banach space. However, it is a $B(\mathfrak{T})$-space, see for instance ([1], Chapter 7, §6, Example 4).

**Remark.** Since the concept of $B(C)$ groups and spaces can be introduced, one can study similar questions in topological groups and topological spaces. For instance, we have the following as a more general result than the usually known one:

**Proposition 4.** *Let $E$ be a Hausdorff locally compact space, $F$ any Hausdorff space and $f$ a continuous almost open mapping of $E$ onto $F$. Then $F$ is also locally compact.*

**Proof.** Let $y=f(x)\in F, x\in E$. Let $U$ be a neighborhood of $x$ such that $\bar{U}$ is compact. Since $f$ is continuous, $V=f(\bar{U})$ is compact and hence closed because $F$ is Hausdorff. Since $f$ is almost open, $\overline{f(U)}$ is a neighborhood of $y$. Clearly $\overline{f(U)}\subset\bar{V}\subset f(\bar{U})$ and thus $\overline{f(U)}$ is a compact neighborhood of $y$.

The usually known form of Proposition 4 is that when $f$ is assumed to be open instead of almost open.

## Reference

[1] T. Husain: The open mapping and closed graph theorems in topological vector spaces. Oxford Mathematical Monographs (1965).