61. *A Generalization of Durszt's Theorem on Unitary ρ-Dilatations*

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In this paper, an operator means a bounded linear operator on a Hilbert space and we use the notations and terminologies of [1].

Let $C_\rho(\rho\geqq 0)$ denote the class of operators T in a Hilbert space $\mathfrak{H}$, whose powers T^n admit a representation

(1) $$T^n=\rho\cdot PU^n \qquad (n=1,2,\cdots)$$

where U is a unitary operator in some Hilbert space K containing $\mathfrak{H}$ as a subspace and P denotes the projection of $\mathfrak{K}$ onto $\mathfrak{H}$. The following theorems were proved by B.Sz-Nagy and C. Foias in [1].

Theorem A. *An operator T in $\mathfrak{H}$ belongs to the class C_ρ if and only if it satisfies the following conditions:*

(I_ρ) $$\|h\|^2-2\Big(1-\frac{1}{\rho}\Big)\operatorname{Re}(zTh,h)+\Big(1-\frac{2}{\rho}\Big)\|zTh\|^2\geqq 0$$

for $h\in\mathfrak{H}$ and $|z|\leqq 1$.

(II) *The spectrum of T lies in the closed unit disk.*

Theorem B. *C_ρ is a non-decreasing function of ρ in the sense that*

$$C_{\rho_1}\subset C_{\rho_2} \qquad \textit{if } 0\leqq\rho_1<\rho_2.$$

These theorems were already proved in [1][2]. Meanwhile E. Durszt [2] has given a simple necessary and sufficient condition for a normal T to belong to C_ρ. In this paper we generalize Durszt's theorem for a suitable class of non-normal operators and show some related results.

Definition 1. An operator T is called a normaloid if $\|T\|=\sup_{\|x\|\leqq 1}|(Tx,x)|$ or equivalently, the spectral radius is equal to $\|T\|$ ([3]—[7]).

Theorem 1. *If T is a normaloid, $T\in C_\rho$ if and only if*

$$\|T\|\leqq\begin{cases}\dfrac{\rho}{2-\rho} & \text{if } 0\leqq\rho\leqq 1\\ 1 & \text{if } \rho\geqq 1.\end{cases}$$

Proof. Let $0\leqq\rho\leqq 1$. In this case (I_ρ) is equivalent with

(I'_ρ) $(2-\rho)\|zTh\|^2-2(1-\rho)\operatorname{Re}(zTh,h)-\rho\|h\|^2\leqq 0$ for $h\in\mathfrak{H}$. $|z|\leqq 1$

That is

(I''_ρ) $$(2-\rho)\|Th\|^2\gamma^2-2(1-\rho)|(Th,h)|\gamma\cos\psi-\rho\|h\|^2\leqq 0$$

for $h\in\mathfrak{H}$, $0\leqq\gamma\leqq 1$, where $z=\gamma e^{i\theta}$, $\psi=\varphi+\theta$, φ; argument of (Th, h) or equivalently,

(2) $(2-\rho)\|Th\|^2\gamma^2+2(1-\rho)|(Th,h)|\gamma-\rho\|h\|^2\leqq 0$ for $h\in\mathfrak{H}$, $0\leqq\gamma\leqq 1$.

Since T is a normaloid, (2) is satisfied if and only if

$$(2-\rho)\|T\|^2\gamma^2+2(1-\rho)\|T\|\gamma-\rho\leqq 0 \quad \text{for } 0\leqq\gamma\leqq 1$$

$$(\|T\|\gamma+1)\{(2-\rho)\|T\|\gamma-\rho\}\leqq 0 \quad \text{for } 0\leqq\gamma\leqq 1.$$

Hence

$$\|T\|\gamma\leqq\frac{\rho}{2-\rho} \quad \text{for } 0\leqq\gamma\leqq 1.$$

Consequently,

(3) $$\|T\|\leqq\frac{\rho}{2-\rho}.$$

Therefore (3) is equivalent with (I_ρ) for $0\leqq\rho\leqq 1$ if T is a normaloid. Now for a normaloid T, the spectral radius is equal to $\|T\|$, so (II) is true if and only if $\|T\|\leqq 1$, consequently $T\in C_\rho$ if and only if (3) holds.

If $\rho\geqq 1$, by the same argument (II) holds if and only if $\|T\|\leqq 1$. By the fact that C_1 consists of the contractions exactly and the monotonity of C_ρ given in Theorem B, we have $T\in C_\rho$ for $\rho\geqq 1$ if and only if $\|T\|\leqq 1$. q.e.d.

Since a hyponormal operator, and hence a normal operator is a normaloid ([6][7]), Theorem 1 gives a generalization of Durszt's theorem concerning ρ-dilatations of operators. For a normaloid T, there exists an approximate proper value having the absolute value $\|T\|$, so our theorem may be proved along E. Durszt's method, but our proof seems to be somewhat direct.

Theorem 2. *Let $\mathfrak{N}_1$ be a maximal family of permutable normal operators in C_1 and put $\mathfrak{N}_\rho=\mathfrak{N}_1\cap C_\rho$, then the family $G=\{\mathfrak{N}_\rho, 0\leqq\rho\leqq 1\}$ forms a commutative semi-group with unit $\mathfrak{N}_1$.*

Proof. If T_i belongs to $\mathfrak{N}_{\rho_i}(i=1,2)$ respectively,

then $$\|T_i\|\leqq\frac{\rho_i}{2-\rho_i}(i=1,2)$$

so we get

(4) $$\|T_1T_2\|\leqq\|T_1\|\,\|T_2\|\leqq\frac{\rho_1}{2-\rho_1}\cdot\frac{\rho_2}{2-\rho_2}.$$

Since T_1, T_2 are permutable normal operators, they are double permutable i.e. $T_1T_2^*=T_2^*T_1$, $T_1^*T_2=T_2T_1^*$, so T_1T_2 is normal ([8][9]), consequently by (4)

$$T_1T_2\in\mathfrak{N}\frac{\rho_1\rho_2}{1+(1-\rho_1)(1-\rho_2)}.$$

We notice that

$$\frac{\rho_1\rho_2}{1+(1-\rho_1)(1-\rho_2)}\leqq\rho_1\rho_2\leqq\text{Min}\,[\rho_1,\rho_2]\leqq 1$$

Hence we may define a product of $\mathfrak{N}_{\rho_1}, \mathfrak{N}_{\rho_2}$ by

$$\mathfrak{N}_{\rho_1}\cdot\mathfrak{N}_{\rho_2}=\mathfrak{N}_{\frac{\rho_1\rho_2}{1+(1-\rho_1)(1-\rho_2)}}.$$

This product clearly satisfies the commutative law and associative law and has $\mathfrak{N}_1$ as a unit. That is

$$\mathfrak{N}_{\rho_1}\cdot\mathfrak{N}_{\rho_2}=\mathfrak{N}_{\rho_2}\cdot\mathfrak{N}_{\rho_1} \qquad \mathfrak{N}_{\rho_1}\cdot(\mathfrak{N}_{\rho_2}\cdot\mathfrak{N}_{\rho_3})=(\mathfrak{N}_{\rho_1}\cdot\mathfrak{N}_{\rho_2})\mathfrak{N}_{\rho_3}$$
$$\mathfrak{N}_1\cdot\mathfrak{N}_\rho=\mathfrak{N}_\rho\cdot\mathfrak{N}_1=\mathfrak{N}_\rho.$$

It is evident that $\mathfrak{N}_\rho(\rho\neq 1)$ has no inverse element.

Denote by A the class of the functions of a complex variable such that

$$v(z)=\sum_{n=1}^{\infty}a_nz^n \quad \text{with} \quad \sum_{n=1}^{\infty}a_n\leqq 1, \qquad a_i\geqq 0 \text{ for all } i.$$

Theorem 3. *Let $u(z)\in A$, and $\mathfrak{N}_\rho$ be the class of Theorem 2. Then $T_i\in\mathfrak{N}_{\rho_i}$ $(i=1,2)$ implies $u(T_1T_2)\in\mathfrak{N}_m$ and*

$$\mathfrak{N}_m\subset\mathfrak{N}_{\rho_1}\cdot\mathfrak{N}_{\rho_2}\subset\mathfrak{N}_{\rho_1},\mathfrak{N}_{\rho_2} \quad \textit{where} \quad m=\frac{2u\left(\frac{\rho_1}{2-\rho_1}\cdot\frac{\rho_2}{2-\rho_2}\right)}{1+u\left(\frac{\rho_1}{2-\rho_1}\cdot\frac{\rho_2}{2-\rho_2}\right)}.$$

Proof. By the same reason in the proof of Theorem 2, $u(T_1T_2)=\sum_1^\infty a_n(T_1T_2)^n$ is a normal operator, so

$$\begin{aligned}\|u(T_1T_2)\|&\leqq\sum_1^\infty|a_n|\,\|(T_1T_2)^n\|=\sum_1^\infty a_n\|T_1T_2\|^n\\&\leqq\sum_1^\infty a_n\left(\frac{\rho_1}{2-\rho_1}\cdot\frac{\rho_2}{2-\rho_2}\right)^n=u\left(\frac{\rho_1}{2-\rho_1}\cdot\frac{\rho_2}{2-\rho_2}\right)\leqq 1\end{aligned}$$

by Theorem 1,

$$u(T_1T_2)\in\mathfrak{N}_m \quad \text{where} \quad m=\frac{2u\left(\frac{\rho_1}{2-\rho_1}\cdot\frac{\rho_2}{2-\rho_2}\right)}{1+u\left(\frac{\rho_1}{2-\rho_1}\cdot\frac{\rho_2}{2-\rho_2}\right)}.$$

Moreover, let

$$l=\frac{\rho_1\rho_2}{1+(1-\rho_1)(1-\rho_2)}, \qquad p=\frac{\rho_1}{2-\rho_1}\cdot\frac{\rho_2}{2-\rho_2},$$

so

$$l-m=\frac{l+lu(p)-2u(p)}{1+u(p)}=\frac{l-(2-l)u(p)}{1+u(p)}.$$

The denominator of right hand is always positive and numerator is equal to

$$\begin{aligned}l-(2-l)u(p)&=l-\frac{2}{1+p}\sum_1^\infty a_np^n=l-\frac{2p}{1+p}\sum_1^\infty a_np^{n-1}=l-l\sum_1^\infty a_np^{n-1}\\&\geqq l-l\sum_1^\infty a_n\geqq 0 \quad \text{therefore} \quad l\geqq m.\end{aligned}$$

By virtue of Theorem B,

$$\mathfrak{N}_m\subset\mathfrak{N}_{\rho_1}\cdot\mathfrak{N}_{\rho_2}\subset\mathfrak{N}_{\rho_1},\mathfrak{N}_{\rho_2}. \qquad \text{q.e.d.}$$

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