60. *Unions of Strongly Paracompact Spaces*

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As is known, a space that is a union of two closed strongly paracompact subspaces need not be strongly paracompact (see [7]). In our previous note ([8]; Theorem 2), we have proved the following theorem:

Let X be a regular T_1-space with a locally finite and star countable closed covering $\{F_\alpha \mid \alpha \in A\}$ such that $Fr(F_\alpha)$ has the locally Lindelöf property and F_α is strongly paracompact for any $\alpha \in A$, then X is itself strongly paracompact.

The main purpose of this note is to omit the hypothesis of the star countability from the above theorem. As the terminologies and notations in this note, refer to our previous note [8].

§ 1. The main theorem. In this section, we shall give the proof of the following theorem:

Theorem 1. *Let $\mathfrak{F}'=\{F'_\alpha \mid \alpha \in A\}$ be a locally finite closed covering of a regular T_1-space X such that $Fr(F'_\alpha)$*[1] *has the locally Lindelöf property for any $\alpha \in A$. Then a necessary and sufficient condition that X be strongly paracompact is that F'_α is strongly paracompact for any $\alpha \in A$.*

Proof. The necessity is obvious and so we shall prove the sufficiency. It is obvious that X is paracompact ([2]; Theorem 1). Now let A be well ordered and $F_0=F'_0$, $F_\alpha=\overline{F'_\alpha-\bigcup_{\beta<\alpha} F'_\beta}$ for each $\alpha>0$ and $Q=\bigcup_{\alpha\neq\beta}(F_\alpha \cap F_\beta)$. Then, similarly as the proof of ([8]; Theorem 1), we can show that $\mathfrak{F}=\{F_\alpha \mid \alpha \in A\}$ is a locally finite closed covering of X and Q is paracompact closed subspace with the locally Lindelöf property (see [8]; Lemma 1 and Lemma 2). Therefore we can get the discrete covering $\mathfrak{G}=\{G_\lambda \mid \lambda \in \Lambda\}$ of Q such that G_λ has the Lindelöf property for any $\lambda \in \Lambda$ by ([4]; Lemma 2.5).

In order to show the strong paracompactness of X, let $\mathfrak{W}$ be an arbitrary open covering of X, and then we shall show that $\mathfrak{W}$ has a star countable open covering of X as a refinement. At first, by the facts that $\mathfrak{G}$ is a discrete collection in the closed subspace Q and $\mathfrak{F}$ is locally finite, there exists a locally finite open covering

1) $Fr(F)$ will denote the boundary of F in X, i.e. $Fr(F)=\overline{F}\cap(\overline{X-F})$.

$\mathfrak{U}$ of X as a star refinement of $\mathfrak{W}$ such that each member of $\mathfrak{U}$ intersects at most one element of $\mathfrak{G}$ and intersects only finitely many elements of $\mathfrak{F}$ (see [6]; Theorem 2).

Next, let $\mathfrak{U}_\alpha = \mathfrak{U} \wedge F_\alpha$[2] for each $\alpha \in A$, and so $\mathfrak{U}_\alpha$ has a star finite open covering $\mathfrak{Q}_\alpha$ of F_α as a refinement by the strong paracompactness of F_α for each $\alpha \in A$. From the Lindelöf property of $F_\alpha \cap G_\lambda$, we can get a countable subcollection $\mathfrak{Q}_{\alpha\lambda}$ of $\mathfrak{Q}_\alpha$ such that $\mathfrak{Q}_{\alpha\lambda}$ is a covering of $F_\alpha \cap G_\lambda$, where we may assume that each member of $\mathfrak{Q}_{\alpha\lambda}$ intersects $F_\alpha \cap G_\lambda$ and so $\{\mathfrak{Q}_{\alpha\lambda} \mid \lambda\}$ is mutually disjoint for each $\alpha \in A$.

Still more, for each λ, let

$$\mathfrak{Q}'_\lambda = \left\{ \bigcup_{j=1}^{n} V^{\alpha_j \lambda}_{l(\alpha_j)} \,\middle|\, \begin{array}{l} V^{\alpha_j\lambda}_{l(\alpha_j)} \in \mathfrak{Q}_{\alpha_j \lambda} \text{ for } j=1,\cdots,n;\ \bigcap_{j=1}^{n} V^{\alpha_j\lambda}_{l(\alpha_j)} \neq \emptyset; \\ l(\alpha_j)=1,2,\cdots \text{for } j=1,\cdots,n; \{\alpha_1,\cdots,\alpha_n\}\text{: finite subset of } A \end{array} \right\}$$

and

$$\mathfrak{Q}_\lambda = \{\mathrm{Int}(V)^{3)} \mid V \in \mathfrak{Q}'_\lambda\}.$$

Then $\mathfrak{Q}_\lambda$ is a covering of $\mathfrak{G}_\lambda$ (see [8]; Proof of Theorem 1), and furthermore we shall show the star countability of $\mathfrak{Q}'_\lambda$ and hence of $\mathfrak{Q}_\lambda$.

For this purpose, let α_0 be any element of A and $V^{\alpha_0\lambda}$ be any member of $\mathfrak{Q}_{\alpha_0\lambda}$, and then it is sufficient to prove that $V^{\alpha_0\lambda}$ intersects at most only countably many elements of $\mathfrak{Q}'_\lambda$. Since $\mathfrak{Q}_\alpha$ is a refinement of $\mathfrak{U}$ for any $\alpha \in A$, $V^{\alpha_0\lambda}$ intersects only finitely many elements of $\{F_\alpha \mid \alpha \in A\}$ and denote all of such indices by $\{\alpha_1, \cdots, \alpha_n\}$. Then, for the countability of $\mathfrak{Q}_{\alpha_i\lambda}$ for $i=1,\cdots,n$, $\bigcup_{i=1}^{n} \mathfrak{Q}^*_{\alpha_i\lambda}$[4] intersects only countably many elements of $\{F_\alpha\}$ and hence denote all of such indices by $\{\beta_1, \beta_2, \cdots\}$. Therefore $\mathfrak{Q}''_\lambda = \left\{ \bigcup_{j=1}^{m} V^{\beta_{ij}\lambda}_{l(\beta_{ij})} \,\middle|\, V^{\beta_{ij}\lambda}_{l(\beta_{ij})} \in \mathfrak{Q}_{\beta_{ij}\lambda}; l(\beta_{i_j}) = 1, 2, \cdots \text{ for } j=1,\cdots,m; \{\beta_{i_1},\cdots,\beta_{i_m}\}\text{: finite subset of } \{\beta_1,\beta_2,\cdots\} \right\}$ is countable and it is easily seen that $V^{\alpha_0\lambda}_{i_0}$ intersects at most only elements of $\mathfrak{Q}''_\lambda$. This fact shows the star countability of $\mathfrak{Q}'_\lambda$ and hence of $\mathfrak{Q}_\lambda$ for each $\lambda \in \Lambda$.

Lastly let

$$\mathfrak{H}_\alpha = \left\{ V - Q \,\middle|\, V \in \mathfrak{Q}_\alpha - \bigcup_\lambda \mathfrak{Q}_{\alpha\lambda} \right\} \qquad \text{for each } \alpha \in A,$$

and

$$\mathfrak{H} = \left(\bigcup_\alpha \mathfrak{H}_\alpha \right) \cup \left(\bigcup_\lambda \mathfrak{Q}_\lambda \right).$$

Then we shall show that this collection $\mathfrak{H}$ is a star countable open covering of X and a refinement of $\mathfrak{W}$.

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2) Let $\mathfrak{A}$ be a collection of subsets of X and, Y be a subset of X, then $\mathfrak{A} \wedge F$ will denote the collection $\{A \cap Y \mid A \in \mathfrak{A}\}$.

3) $\mathrm{Int}(V)$ will denote the interior of V in X.

4) For the collection $\mathfrak{A}$ of subsets of X, $\mathfrak{A}^*$ will denote the set $\cup\{A \mid A \in \mathfrak{A}\}$.

(1) *$\mathfrak{H}$ is an open covering of X and a refinement of $\mathfrak{W}$.* This is quite similarly proved as our previous note ([8]; Proof of Theorem 1).

(2) *$\mathfrak{H}$ is star countable.*

(2.1) Let α_0 be an arbitrary element of A and $V-Q$ be an arbitrary element of $\mathfrak{H}_{\alpha_0}$, where $V\in\mathfrak{Q}_{\alpha_0}-\bigcup_{\lambda}\mathfrak{Q}_{\alpha_0\lambda}$, and then we shall show that *$V-Q$ intersects only countably many elements of $\mathfrak{H}$.*

(2.1.1) *$\mathfrak{H}^*_{\alpha_0}\cap\mathfrak{H}^*_{\alpha}=\emptyset$ for any $\alpha\neq\alpha_0$.* This is obvious, by the definitions of $\{\mathfrak{H}_\alpha\mid\alpha\in A\}$ and Q.

(2.1.2) *All the indices $\{\lambda\mid\lambda\in\Lambda\}$ such that $(V-Q)\cap(\mathfrak{Q}'_\lambda)^*\neq\emptyset$ is finite.* Suppose $(V-Q)\cap V_0\neq\emptyset$ for $V_0=\bigcup_{j=1}^{n}V^{\alpha_j\lambda}_{l(\alpha_j)}\in\mathfrak{Q}'_\lambda$ where $V^{\alpha_j\lambda}_{l(\alpha_j)}\in\mathfrak{Q}_{\alpha_j\lambda}$ and $\bigcap_{j=1}^{n}V^{\alpha_j\lambda}_{l(\alpha_j)}\neq\emptyset$. Since $V-Q\subset\mathrm{Int}(F_{\alpha_0})=\{y\mid y\notin\bigcup_{\alpha\neq\alpha_0}F_\alpha\}$, we have $\alpha_0\in\{\alpha_1,\cdots,\alpha_n\}$ and $(V-Q)\cap V^{\alpha_0\lambda}_{l(\alpha_0)}\neq\emptyset$, and so $V\cap V^{\alpha_0\lambda}_{l(\alpha_0)}\neq\emptyset$. On the other hand, V, $V^{\alpha_0\lambda}_{l(\alpha_0)}\in\mathfrak{Q}_{\alpha_0\lambda}\subset\mathfrak{Q}_{\alpha_0}$ and furthermore $\mathfrak{Q}_{\alpha_0}$ is star finite, and hence V intersects only finitely many elements of $\bigcup_{\lambda}\mathfrak{Q}_{\alpha_0\lambda}$. If we denote them by $V^{\alpha_0\lambda_1},\cdots,V^{\alpha_0\lambda_m}$, where $V^{\alpha_0\lambda_i}\in\mathfrak{Q}_{\alpha_0\lambda_i}$ for each $i=1,\cdots,m$, then we have $\lambda\in\{\lambda_1,\cdots,\lambda_m\}$ by the mutual disjointedness of $\{\mathfrak{Q}_{\alpha_0\lambda}\mid\lambda\in\Lambda\}$.

(2.1.3) *$V-Q$ intersects only countably many elements of $\mathfrak{Q}'_\lambda$ for any λ.* In fact, since $\mathfrak{Q}^*_{\alpha_0\lambda}$ intersects only countably many elements of $\{F_\alpha\mid\alpha\in A\}$ (we denote them by $\{\beta_1,\beta_2,\cdots\}$), $\alpha_0\in\{\alpha_1,\cdots,\alpha_n\}$ and $\bigcap_{j=1}^{n}V^{\alpha_j\lambda}_{l(\alpha_j)}\neq\emptyset$, so we have $\{\alpha_1,\cdots,\alpha_n\}\subset\{\beta_1,\beta_2,\cdots\}$. This means: If $(V-Q)\cap V_0\neq\emptyset$, then $V_0\in\mathfrak{Q}'''_\lambda$, where

$$\mathfrak{Q}'''_\lambda=\left\{\bigcup_{t=1}^{k}V^{\beta_{i_t}\lambda}_{l(\beta_{i_t})}\;\middle|\;\begin{array}{l}V^{\beta_{i_t}\lambda}_{l(\beta_{i_t})}\in\mathfrak{Q}_{\beta_{i_t}\lambda};\ l(\beta_{i_t})=1,2,\cdots;\ \{\beta_{i_1},\cdots,\beta_{i_p}\}:\\ \text{finite subset of }\{\beta_1,\beta_2,\cdots\}\end{array}\right\}.$$

Therefore $V-Q$ intersects only countably many elements of $\mathfrak{Q}'_\lambda$ from the countability of $\mathfrak{Q}'''_\lambda$.

(2.1.4) *$\mathfrak{H}_{\alpha_0}$ is star countable.* This is obvious by the star countability of $\mathfrak{Q}_{\alpha_0}$.

From the assertions (2.1.1), (2.1.2), (2.1.3), and (2.1.4), we can get (2.1).

(2.2) Let λ_0 be an arbitrary index of Λ and $V_0=\bigcup_{j=0}^{n}V^{\alpha_j\lambda_0}_{l(\alpha_j)}$ be any element of $\mathfrak{Q}'_{\lambda_0}$ (where $V^{\alpha_j\lambda_0}_{l(\alpha_j)}\in\mathfrak{Q}_{\alpha_j\lambda_0}$ for $j=0,1,\cdots,n$), and then we shall show that V_0 intersects only countably many elements of $\mathfrak{H}$. For this purpose, it is sufficient to show that *$V^{\alpha_0\lambda_0}\in\mathfrak{Q}_{\alpha_0\lambda_0}$ intersects only countably many elements of $\mathfrak{H}$.*

(2.2.1) *$\mathfrak{Q}'_{\lambda_0}$ is star countable.* This is already shown.

(2.2.2) *All the indices $\{\lambda'\mid\lambda'\neq\lambda_0\}$ that $V^{\alpha_0\lambda_0}$ intersects $\mathfrak{Q}^*_{\alpha_0\lambda'}$* is countable. This is evident by the mutual disjointedness of $\{\mathfrak{Q}_{\alpha_0\lambda}\mid\lambda\}$ and we denote their indices by $\{\lambda_1,\lambda_2,\cdots\}$.

(2.2.3) *If* $\lambda\neq\lambda_0$ *and* $\beta\neq\alpha_0$, *then* $V^{\alpha_0\lambda_0}\cap V^{\beta\lambda}=\emptyset$ *for any* $V^{\beta\lambda}\in\mathfrak{Q}_{\beta\lambda}$. Suppose $x\in V^{\alpha_0\lambda_0}\cap V^{\beta\lambda}$, then $x\in F_{\alpha_0}\cap F_\beta\subset Q=\bigcup_\lambda G_\lambda$ and so $x\in G_{\lambda'}$ for some $\lambda'\in\Lambda$. On the other hand, since $V^{\alpha_0\lambda_0}\in\mathfrak{Q}_{\alpha_0\lambda_0}$, hence $V^{\alpha_0\lambda_0}\cap G_\lambda\neq\emptyset$ and so we have $\lambda'=\lambda_0$ by $x\in V^{\alpha_0\lambda_0}\cap G'_\lambda$. Similarly we have $\lambda'=\lambda$. This contradicts to $\lambda\neq\lambda_0$.

(2.2.4) $V^{\alpha_0\lambda_0}$ *intersects only countably many elements of* $\mathfrak{Q}'_{\lambda'}$ *for any* $\lambda'\neq\lambda_0$. Suppose $V'_0=\bigcup_{i=1}^m\{V^{\beta_i\lambda'}_{l(\beta_i)}\in\mathfrak{Q}_{\beta_i\lambda'}\}\in\mathfrak{Q}'_{\lambda'}$ and $V'_0\cap V^{\alpha_0\lambda_0}\neq\emptyset$. Then we have $\lambda'=\lambda_{i_0}$ for some i_0, and $\alpha_0\in\{\beta_1,\cdots,\beta_m\}$ from (2.2.2) and (2.2.3). Since $\{\mathfrak{Q}_{\alpha_0\lambda_i}\mid i=1,2,\cdots\}$ is countable, $\bigcup_{i=1}^\infty\mathfrak{Q}^*_{\alpha_0\lambda_i}$ intersects only countably many elements of $\{F_\alpha\mid\alpha\in A\}$ and we denote them by $\{F_{\gamma_1},F_{\gamma_2},\cdots\}$.

Furthermore, by the definition of $\mathfrak{Q}'_{\lambda i_0}$, we have $\bigcap_{i=1}^m V^{\beta_i\lambda i_0}_{l(\beta_i)}\neq\emptyset$. This means that if $V^{\alpha_0\lambda_0}\cap\left(\bigcup_{i=1}^m V^{\beta_i\lambda'}_{l(\beta_i)}\right)\neq\emptyset$, then $\lambda'\in\{\lambda_1,\lambda_2,\cdots\}$ and $\beta_i\in\{\gamma_j\mid j=1,2,\cdots\}$ for $i=1,\cdots,m$. Therefore, if $V^{\alpha_0\lambda_0}\cap V'_0\neq\emptyset$ and $V'_0\in\mathfrak{Q}'_{\lambda'}$, then $V'_0\in\cup\{\mathfrak{Q}_{\gamma_j\lambda_i}\mid i,j=1,2,\cdots\}=\mathfrak{Q}''''_{\lambda_0}$. From the countability of $\mathfrak{Q}''''_{\lambda_0}$, we can get the assertion (2.2.4).

(2.2.5) Let α be any element of $A-\{\alpha_0\}$ and $V-Q$ be any member of $\mathfrak{H}_\alpha$ where $V\in\mathfrak{Q}_\alpha-\bigcup_\lambda\mathfrak{Q}_{\alpha\lambda}$ and then $(V-Q)\cap V^{\alpha_0\lambda_0}=\emptyset$. This is obvious from the facts that $V\subset F_\alpha$ and $V^{\alpha_0\lambda_0}\in F_{\alpha_0}$.

(2.2.6) $V^{\alpha_0\lambda_0}$ *intersects only countably many elements of* $\mathfrak{H}_{\alpha_0}$. This is obvious from the star finiteness of $\mathfrak{Q}_{\alpha_0}$.

Therefore we can see that $V^{\alpha_0\lambda_0}$ intersects only countably many elements of $\bigcup_\lambda\mathfrak{Q}'_\lambda$ (resp. $\bigcup_\alpha\mathfrak{H}_\alpha$) from the assertions (2.2.1) and (2.2.4) (resp. (2.2.5) and (2.2.6)), and hence we get the assertion (2.2).

From (1) and (2), $\mathfrak{W}$ has a star countable open covering $\mathfrak{H}$ of a regular T_1-space X as a refinement and so, X is strongly paracompact by Yu. M. Smirnov ([5]; Theorem 1). This completes the proof of Theorem 1.

Remark 1. Theorem 1 is a generalization of ([1]; Theorem 2), ([7]; Proposition 5) and ([8]; Theorem 1, Theorem 2 and Theorem 3).

In Theorem 1, we cannot omit the local finiteness of $\{F'_\alpha\mid\alpha\}$ or the local Lindelöf property of $Fr(F'_\alpha)$.

§2. Applications. In this section, by use of Theorem 1, we get some propositions concerning the strong paracompactness of the spaces with some kinds of open coverings.

Theorem 2. *Let* $\{U_\alpha\mid\alpha\in A\}$ *be a locally finite open covering of a normal* T_1*-space* X *such that* U_α *is strongly paracompact for any* $\alpha\in A$. *If* X *has the locally Lindelöf property at* $\bigcup_\alpha Fr(U_\alpha)$ *(see* [8]*), then* X *is strongly paracompact.*

Proof. By the normality of X and the local finiteness of

$\{U_\alpha \mid \alpha \in A\}$, there exists an open covering $\{V_\alpha \mid \alpha \in A\}$ of X such that $\bar{V}_\alpha \subset U_\alpha$ for any $\alpha \in A$. Since X has the locally Lindelöf property at $\bigcup_\alpha Fr(U_\alpha)$, there exists an open neighborhood $V(x)$ of x with the Lindelöf closure such that $V(x) \cap \bar{V}_\alpha = \emptyset$ for each $x \in Fr(U_\alpha)$.

Now, let $H_\alpha = \bar{U}_\alpha - \bigcup_{x \in Fr(U_\alpha)} V(x)$ for each $\alpha \in A$, then H_α is closed in X and $\bar{V}_\alpha \subset H_\alpha \subset U_\alpha$. Therefore there exists an open set G_α such that $H_\alpha \subset G_\alpha \subset \bar{G}_\alpha \subset U_\alpha$. Then it is easily seen that $Fr(G_\alpha) \subset \bigcup_{x \in Fr(U_\alpha)} V(x)$ for any α, and so $Fr(G_\alpha)$ has the locally Lindelöf property for any $\alpha \in A$.

Thus we can get the locally finite closed covering $\{\bar{G}_\alpha \mid \alpha \in A\}$ of X such that $\bar{G}_\alpha$ is strongly paracompact and $Fr(G_\alpha)$ has the locally Lindelöf property for any $\alpha \in A$. Consequently X is strongly paracompact by Theorem 1.

Theorem 3. *Let $\{U_\alpha\}$ be a locally finite open covering of a regular T_1-space X such that U_α is strongly paracompact for any α and $Fr(U_\alpha)$ has the Lindelöf property. If X has the locally Lindelöf property at $\bigcup_\alpha Fr(U_\alpha)$, then X is strongly paracompact.*

Proof. By A. Okuyama ([3]; Theorem 1), X is normal, and hence X is strongly paracompact by Theorem 2.

Theorem 4. *Let $\{G_\alpha \mid \alpha \in A\}$ be a locally finite open covering of a normal T_1-space X and there exists an open set H_α such that $Fr(G_\alpha) \subset H_\alpha \subset \bar{H}_\alpha$ for each $\alpha \in A$ and $\bigcup_\alpha (\overline{G_\alpha - H_\alpha}) = X$. If $Fr(H_\alpha)$ has the locally Lindelöf property and G_α is strongly paracompact for any $\alpha \in A$, then X is strongly paracompact.*

Proof. For each $\alpha \in A$, let $F_\alpha = \overline{G_\alpha - H_\alpha}$, then $\{F_\alpha \mid \alpha \in A\}$ is a locally finite closed covering of X. Furthermore, it is easily seen that $F_\alpha \subset G_\alpha$ and $Fr(F_\alpha) \subset Fr(H_\alpha)$ for any $\alpha \in A$. Therefore, since $\{F_\alpha\}$ clearly satisfies the hypotheses in Theorem 1, this completes the proof.

Remark 2. Theorem 2 is a generalization in our previous note ([8]; Theorem 4) and, Theorem 3 is a generalization of ([1]; Theorem 5) from the point of view of obtaining only the strong paracompactness of a space.

In Theorem 4, we assumed the normality of X instead of regularity of X, but we can omit or weaken some conditions in A. Okuyama's Theorem ([3]; Theorem 2).

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