59. *Infinite Product of Ergodic Flows with Pure Point Spectra*

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J. von Neumann has shown that two ergodic flows with the same pure point spectrum are mutually metrically equivalent, and that they are isomorphic to the canonical flow on a compact Abelian group. While non-ergodic flows with the same pure point spectrum are not always mutually metrically equivalent.

In this paper, we shall show that, for a certain class of non-ergodic flows with pure point spectra, the spectral type determines the flow metrically. Flows of this class often appear as flows induced by stationary stochastic processes and as transversal flows of automorphisms.

1. Canonical flows on compact Abelian groups. Let G be a separable compact Abelian group, m be the normalized Haar measure of G and $\mathfrak{M}$ be the minimum complete σ-field generated by all open subsets of G. Then $(G, \mathfrak{M}, m)$ is a Lebesgue space in the sense of V. A. Rohlin [4]. Let $\{\alpha^t\}$ be a one-parameter subgroup of G. Then the flow $\{S_t\}$ on G is defined by

(1,1) $$S_t g = \alpha^t g \qquad \text{for } g \in G.$$

Definition 1. We call $\{S_t\}$ the G-flow induced by $\{\alpha^t\}$.

The G-flow $\{S_t\}$ is measurable and it is ergodic if and only if $H = \{\alpha^t; -\infty < t < \infty\}$ is dense in G.

If Λ is a countable subgroup of the additive group R of real numbers, then its character group G is a separable compact Abelian group and there exists a one-parameter subgroup $\{\alpha^t\}$ of G, such that

(1,2) $$\alpha^t(\lambda) = \exp[it\lambda] \qquad \text{for } \lambda \in \Lambda.$$

Let $\{S_t\}$ be the G-flow induced by the $\{\alpha^t\}$. Then $\{S_t\}$ is an ergodic flow with the pure point spectrum Λ. Conversely, for any ergodic measurable flow on a Lebesgue space, the discrete part of the spectrum forms a countable subgroup of R.

Definition 2. We call the G-flow induced by this $\{\alpha^t\}$ a canonical flow on G.

Theorem 1. (*J. von Neumann*). *Let $\{T_t\}$ be an ergodic measurable flow on a Lebesgue space with a pure point spectrum Λ. Let $\{S_t\}$ be a canonical flow on G which is the character group of Λ. Then $\{T_t\}$ and $\{S_t\}$ are isomorphic to each other.*

2. Product of canonical flows. Let $\Lambda_j; j=1, 2, 3, \cdots, N$ $(1 \leq N \leq \infty)$ be a system of countable subgroups of R, and G_j be of the character group of Λ_j. We denote by Γ the direct product of the discrete groups $\Lambda_j; j=1, 2, 3, \cdots, N$, and by Ω the direct product of the compact groups $G_j; j=1, 2, 3, \cdots, N$: $\Gamma=\{\boldsymbol{\lambda}=(\lambda_1, \lambda_2, \cdots)\}$ and $\Omega=\{w=(g_1, g_2, \cdots)\}$. Then Ω is the character group of Γ. Let Λ be the countable subgroup of R which is generated by $\Lambda_j; j=1, 2, \cdots, N$, and G be the character group of Λ. Let θ be the natural homomorphism of Γ onto Λ such that

$$(2,1) \qquad \theta(\boldsymbol{\lambda})=\sum_j \lambda_j \qquad \text{for } \boldsymbol{\lambda}=(\lambda_1, \lambda_2, \cdots).$$

The kernel of θ and the character group of it are denoted by Z and K, respectively. Then Ω, G, and K are separable compact Abelian groups. Let P, m, m_0, and $m_j; j=1, 2, \cdots$ be normalized Haar measures of Ω, G, K, and $G_j; j=1, 2, \cdots$, respectively. The measure P turns out to be the direct product of the m_j. Let $\{\alpha_j^t\}$ be the one-parameter subgroup of G_j determined by

$$(2,2) \qquad \alpha_j^t(\lambda_j)=\exp[it\lambda_j] \qquad \text{for } \lambda_j \in \Lambda_j,$$

and $\{T_t\}$ be the G-flow determined by

$$(2,3) \qquad T_t w=\alpha^t w=(\alpha_1^t g_1, \alpha_2^t g_2, \cdots) \qquad \text{for } w \in \Omega.$$

Under these notations, we state our theorems.

Theorem 2. *The flow $\{T_t\}$ defined by* (2,3) *has the following properties,*

1) *$\{T_t\}$ has the pure point spectrum Λ,*
2) *if $Z=\{0\}$, then $\{T_t\}$ is ergodic,*
3) *if $Z\neq\{0\}$, then $\{T_t\}$ is not ergodic and each point $\lambda \in \Lambda$ has infinite multiplicity.*

Proof. Since $\{g_j(\lambda_j)=\lambda_j(g_j); \lambda_j \in \Lambda_j\}$ is a complete orthonormal system of $L^2(G_j, m_j)$, $\{f_{\boldsymbol{\lambda}}(w)=g_1(\lambda_1)g_2(\lambda_2)\cdots g_n(\lambda_n); \boldsymbol{\lambda}=(\lambda_1, \cdots, \lambda_n, 0, 0, \cdots) \in \Gamma\}$ forms a complete orthonormal system of $L^2(\Omega, P)$. We also have

$$(2,3) \qquad f_{\boldsymbol{\lambda}}(T_t w)=(\alpha_1^t g_1)(\lambda_1)\cdots(\alpha_n^t g_n)(\lambda_n) = \exp[it\theta(\boldsymbol{\lambda})] f_{\boldsymbol{\lambda}}(w).$$

Hence 1) is proved. Noting that any non-trivial subgroup of Z is a infinite group, 2) and 3) are easily derived from (2,3).

Lemma 1. *Let $(\Omega, \mathcal{B}, P)$ be a Lebesgue space and $\{T_t\}$ be a measurable flow on it. Let ζ be the measurable partition of Ω into the ergodic parts for $\{T_t\}$. Suppose that there exist two orthonormal systems of $L^2(\Omega, P)$, $\{f_\lambda; \lambda \in \Lambda\}$ and $\{\psi_n\}$ such that*

i) *Λ is a countable subgroup of R,*

ii) *there exists a system of $\mathcal{B}(\zeta)$-measurable functions $\{h(\lambda, \mu); \lambda, \mu \in \Lambda\}$ satisfying*

$$(2,4) \qquad f_\lambda \cdot f_\mu = h(\lambda, \mu) f_{\lambda+\mu},$$

iii) $f_\lambda(T_t w)=\exp[it\lambda]f_\lambda(w)$, $|f_\lambda(w)|\equiv 1$,

iv) $\psi_n(T_t w)=\psi_n(w)$, $|\psi_n|\equiv 1$,

v) $\{\psi_n\}$ *is a complete orthonormal system of* $L^2(\Omega, \mathscr{B}(\zeta), P_\zeta)$,

vi) $\{f_\lambda\psi_n; \lambda\in\Lambda, n\}$ *is a complete orthonormal system of* $L^2(\Omega, \mathscr{B}, P)$.

Then there exists a system of $\mathscr{B}(\zeta)$-measurable functions $\{\gamma(\lambda); \lambda\in\Lambda\}$ such that $\{\varphi_\lambda=\gamma(\lambda)f_\lambda; \lambda\in\Lambda\}$ and $\{\psi_n\}$ satisfy the conditions iii)$\sim$vi) and that

ii)′ $\varphi_\lambda\cdot\varphi_\mu=\varphi_{\lambda+\mu}$.

Lemma 2. *Assume the same conditions as in Lemma* 1, *and let* ξ *be the partion induced by* $\{\varphi_\lambda; \lambda\in\Lambda\}$. *Then* ζ *and* ξ *are* $\{T_t\}$-*invariant and they are mutually independent.*

Lemma 3. *Under the same conditions as in Lemma* 2, $\{T_t\}$ *is isomorphic to the product flow of the factor flows* $\{T_t^\zeta\}$ *and* $\{T_t^\xi\}$.

Lemma 4. *Under the same conditions as in Lemma* 2, $\{T_t^\xi\}$ *is isomorphic to the canonical flow* $\{S_t\}$ *on G which is the character group of* Λ, *and* $\{T_t^\zeta\}$ *is the identity flow (i.e.* T_t *is the identity of* $\Omega_{/\zeta}$ *for every* t).

By virtue of Lemma 1$\sim$4, we have the following theorem.

Theorem 3. *Under the same conditions as in Theoaem* 2, $\{T_t\}$ *is isomorphic to the flow which is the direct product of the canonical flow* $\{S_t\}$ *on G and the identity flow on K.*

Corollary 1. *Countable direct product of ergodic flows with pure point spectra is either ergodic or isomorphic to the direct product of an ergodic flow and an identity flow on a Lebesgue space without atoms. The metrical type of this product flow is determined by its spectral type.*

Corollary 2. *Let* $\{T_t\}$ *be a flow on a N-dimensional torus* $\boldsymbol{T}^N(1\le N\le\infty)$ *such that*

(2,5) $T_t w=(x_1+\lambda_1 t, x_2+\lambda_2 t, \cdots)$ *for* $w=(x_1, x_2, \cdots)\in\boldsymbol{T}^N$,

and let Λ *be the additive group generated by* $\Gamma=\{\lambda_n\}$. *Then*

1) *if the elements of* Γ *are linearly independent,* $\{T_t\}$ *is an ergodic flow with the pure point spectrum* Λ,

2) *if the elements of* Γ *are linearly dependent,* $\{T_t\}$ *is a non-ergodic flow with the pure point spectrum* Λ, *and each point* $\lambda\in\Lambda$ *has infinite multiplicity. The flow* $\{T_t\}$ *is isomorphic to the product of the canonical flow on G which is the character group of* Λ *and the identity flow on a Lebesgue space without atoms.*

Theorem 4. *Two flows which are induced by multi-dimensional Gaussian stationary processes with pure point spectra are mutually metrically equivalent if and only if they have the same spectral type.*

3. **Product of automorphisms.** We have similar results in

the cases of automorphisms as in the cases of flows.

Let Λ be a countable subgroup of one-dimensional torus $\boldsymbol{T}^1$ and let G be the character group of Λ. Then there exists an element α of G such that

(3,1) $$\alpha(\lambda)=\exp[i\lambda] \qquad \text{for } \lambda\in\Lambda.$$

Now, let T be an automorphism of G such that

(3,2) $$Tg=\alpha g \qquad \text{for } g\in G.$$

Then T is an ergodic automorphism with the pure point spectrum Λ. Conversely, an ergodic automorphism of a Lebesgue space with the pure point spectrum Λ is isomorphic to the automorphism of the type (3,2).

Let Λ_j; $j=1, 2, 3, \cdots$ be a system of countable subgroup of $\boldsymbol{T}^1$, and G_j be the character group of Λ_j and α_j be the element of G_j satisfying (3,2). Let Γ be the direct product of the Λ_j and Ω be the direct product of the G_j. Let Λ be the countable subgroup of $\boldsymbol{T}^1$ generated by the Λ_j and θ be the natural homomorphism of Γ onto Λ such that

(3,3) $$\theta(\boldsymbol{\lambda})=\sum_j\lambda_j \qquad \boldsymbol{\lambda}=(\lambda_1, \lambda_2, \cdots)\in\Gamma$$

and let Z be the kernel of θ and K be the character group of Z. Then an automorphism T of Ω is defined by

(3,4) $$Tw=(\alpha_1g_1, \alpha_2g_2, \cdots) \qquad \text{for } w=(g_1, g_2, \cdots)\in\Omega.$$

Theorem 5. *The automorphism T defined by* (3,4) *has following properties,*

1) *T has the pure point spectrum Λ,*

2) *if $Z=\{0\}$, then T is ergodic,*

3) *if $Z\neq\{0\}$, then T is not ergodic and has uniform multiplicity k which is the cardinal number of Z. The automorphism T is isomorphic to the product of the identity of K and the automorphism of G which is defined by* (3,2).

Two automorphisms of this type are mutually metrically equivalent if and only if they have the same spectral type.

Remark. If Z is a finite set with cardinal number k then K has k atoms with measure $1/k$. If Z is an infinite set, K is a Lebesgue space without atoms.

Corollary 1. Let T be an automorphism of $\boldsymbol{T}^N$ such that

(3,5) $$Tw=(x_1+\lambda_1, x_2+\lambda_2, \cdots) \qquad \text{for } w\in\boldsymbol{T}^N$$

and let Λ be the subgroup of $\boldsymbol{T}^1$ generated by $\Gamma=\{\lambda_n\}$. Then

1) if the elements of Γ are linearly independent, T is an ergodic automorphism with the pure point spectrum Λ,

2) if the elements of Γ are linearly dependent, T is non-ergodic automorphism with the pure point spectrum Λ, and each point $\lambda\in\Lambda$ has infinite multiplicity. The automorphism T is isomorphic to the

product of the identity of a Lebesgue space without atoms and an ergodic automorphism with the pure point spectrum Λ.

Two automorphisms of this type are mutually metrically equivalent if and only if they have the same spectral type.

Theorem 6. *Two automorphisms which are induced by multi-dimensional discrete Gaussian stationary processes with pure point spectra are mutually metrically equivalent if and only if they have the same spectral type.*

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