83. A Criterion for the Separable Axiomatization of Gödel's S_n

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This report is an extension to our papers [2] and [3]. And we use notations and results of them without mentioning.

In this paper, we report a criterion for an axiom scheme to give a separable axiomatic system for $\boldsymbol{S}_n$ by adding it to Dummett's $\boldsymbol{LC}$, and we also report that there is no intermediate axiomatic system between $\boldsymbol{S}_n$ and $\boldsymbol{S}_{n+1}$. Our result is also an extension to that obtained by Hanazawa [1] in the following form, though we do not suppose familiarity with it.

Theorem 1 (*By Hanazawa*). *The system* $\boldsymbol{LI}+A$ *is equivalent to the usual classical system* $\boldsymbol{S}_1$ *if and only if* A *is valid in* $\boldsymbol{S}_1$ *but not in* $\boldsymbol{S}_2$.

Since axiomatic systems are known for $\boldsymbol{S}_n$'s, the validity in $\boldsymbol{S}_n$ is equivalent to the provability in $\boldsymbol{S}_n$. Though Hanazawa does not mention explicitly, the above theorem implies the following

Corollary 2. *There is not an intermediate axiomatic system between* $\boldsymbol{S}_1$ *and* $\boldsymbol{S}_2$.

Further, we have a stronger

Corollary 3. *If* $\boldsymbol{S}_1 \supsetneqq \boldsymbol{L} \subset \boldsymbol{LI}$, *then necessarily* $\boldsymbol{S}_2 \supset \boldsymbol{L}$.

Proof. Suppose that $\boldsymbol{L} \vdash A$ but not that $\boldsymbol{S}_2 \vdash A$. Then $\boldsymbol{S}_1 \supset\subset \boldsymbol{LI}+A$ since $\boldsymbol{S}_1 \vdash A$. On the other hand, we obviously have that $\boldsymbol{L} \supset \boldsymbol{LI}+A$. This is contrary to our hypothesis.

Before we mention our theorem, we remark that the above theorem does not generally hold for $\boldsymbol{S}_n$ and $\boldsymbol{S}_{n+1}$ in the above form. Let us take the formula P_n of Nagata, for example. We know that P_n is valid in $\boldsymbol{S}_n$ but not in $\boldsymbol{S}_{n+1}$, but as we reported in [2], $\boldsymbol{LI}+P_n$ is not equivalent to $\boldsymbol{S}_n$. So we prove a similar theorem in the following form.

Theorem 4. $\boldsymbol{LC}+A \supset\subset \boldsymbol{S}_n$ *if and only if* $\boldsymbol{S}_n \vdash A$ *and not* $\boldsymbol{S}_{n+1} \vdash A$.

Before we prove the theorem, we quote some lemmas from our previous papers without proof.

Lemma 5. Suppose a formula A has k distinct propositional variables at the most. *Then* $\boldsymbol{LC} \vdash A$ *if and only if* $\boldsymbol{S}_{k+1} \vdash A$.

Lemma 6. Suppose that A does not contain the logical operation $-$. *Then, under the hypothesis of the lemma* 5, $\boldsymbol{LC}\vdash A$ *if and only if* $\boldsymbol{S}_k\vdash A$.

Lemma 7. Let be that $\boldsymbol{S}_n\vdash A$ and let f be an assignment of $\boldsymbol{LC}$. *If f satisfies one of the following conditions, then* $f(A)=1$:

(1) $H(f)\leq n-1$.

(2) $H(f)=n$, *and* $V(f)\ni 1$ *or* $\ni\omega$.

(3) $H(f)=n+1$, *and* $V(f)\ni 1$ *and* $\ni\omega$.

Lemma 8. Let A be a formula which does not contain the logical operation $\neg$. Let be that $\boldsymbol{S}_n\vdash A$ and let f be an assignment of $\boldsymbol{LC}$. *If f satisfies one of the following conditions, then* $f(A)=1$:

(1) $H(f)\leq n$.

(2) $H(f)=n+1$, *and* $V(f)\ni 1$.

Lemma 9. Let be that $\boldsymbol{S}_n\vdash A$. *If f is an assignment of* $\boldsymbol{S}_{n+1}$, *then* $f(A)=1$ *or* $f(A)=2$.

Lemma 10. $\boldsymbol{LC}+R_n\supset\subset\boldsymbol{LI}+R_n\supset\subset\boldsymbol{S}_n$, *where*

$$R_n=a_1\vee(a_1\supset a_2)\vee\cdots\vee(a_{n-1}\supset a_n)\vee\neg a_n.$$

Proof of Theorem 4. (i) Suppose that $\boldsymbol{LC}+A\supset\subset\boldsymbol{S}_n$. Then obviously $\boldsymbol{S}_n\vdash A$. If $\boldsymbol{S}_{n+1}\vdash A$, then $\boldsymbol{S}_n\supsetneqq\boldsymbol{S}_{n+1}\supset\boldsymbol{LC}+A$. This is contrary to the hypothesis.

(ii) Suppose that $\boldsymbol{S}_n\vdash A$ but not that $\boldsymbol{S}_{n+1}\vdash A$. Then there is an assignment φ of $\boldsymbol{S}_{n+1}$ such that $\varphi(A)=2$. Without loss of generality, we can suppose that A contains the propositional variables $a_1,\cdots,a_k$ and only those. By the lemma 7, we can suppose that $V(\varphi)\supset\{2,\cdots,n+1\}$. We substitute the propositional variables of A with regard to the assignment φ as follows. First, if $i<j$ and $\varphi(a_i)=\varphi(a_j)$, a_j is substituted by a_i. We repeat this substitution as long as this can be operated. Since $V(\varphi)\supset\{2,\cdots,n+1\}$, there is a propositional variable for which φ assigns the value 2. We suppose that $\varphi(a_i)=2$. Then we substitute those a_j's for which $\varphi(a_j)=1$ or $\varphi(a_j)=\omega$ by $a_i\supset a_i$ or $\neg(a_i\supset a_i)$, respectively. Then the obtained formula contains exactly n propositional variables. And lastly, we do substitution in A so that A contains just $a_1,\cdots,a_n$ and that $\varphi(a_i)<\varphi(a_j)$ if and only if $i<j$. Since $V(\varphi)=\{2,\cdots,n+1\}$, $\varphi(a_i)=i+1$. We call this formula obtained from A by substitution as A^φ. If it is proved that $\boldsymbol{LC}\vdash A^\varphi\supset R_n$, then $\boldsymbol{LC}+A\vdash R_n$ and so $\boldsymbol{LC}+A\supset\boldsymbol{S}_n$. And on the other hand we have obviously that $\boldsymbol{S}_n\supset\boldsymbol{LC}+A$. So $\boldsymbol{LC}+A\supset\subset\boldsymbol{S}_n$ is proved. Hence we only need to prove that $\boldsymbol{LC}\vdash A^\varphi\supset R_n$. Since the formula $A^\varphi\supset R_n$ has n distinct propositional variables, it will be sufficient if we prove that $\boldsymbol{S}_{n+1}\vdash A^\varphi\supset R_n$ by the lemma 5. Let f be an assignment of $\boldsymbol{S}_{n+1}$. If $f(R_n)=1$, then obviously $f(A^\varphi\supset R_n)=1$. Suppose that $f(R_n)\neq 1$.

By the definition of R_n, $f(R_n)\neq 1$ if and only if $f(a_i)=i+1$. Hence $f(A^\varphi)\neq 1$. By the lemma 9, $f(A^\varphi)=f(R_n)=2$. Hence $f(A^\varphi\supset R_n)=1$. Q.e.d.

If A does not contain the logical operation $\neg$, the part (ii) of the above proof can be differently treated as follows. Let R'_n be $a_1\vee(a_1\supset a_2)\vee\cdots\vee(a_{n-1}\supset a_n)\vee(a_n\supset a_{n+1})$. This R'_n is interdeducible with R_n in $\boldsymbol{LI}$. Hence $\boldsymbol{LI}+R'_n\supset\subset\boldsymbol{S}_n$. Let φ be an assignment of $\boldsymbol{S}_{n+1}$ such that $\varphi(A)=2$ as in above. By the lemma 8, $V(\varphi)\supset\{2,\cdots,n+1,n+2\}$, where $n+2$ stands for ω for convenience. Then we do substitution for A just as before only excepting the case of $\varphi(a_j)=\omega$. A^φ only contains the propositional variables $a_1,\cdots,a_{n+1}$. And $f(A^\varphi)\neq 1$ if and only if $f(a_i)<f(a_j)$ for i and j such that $i<j$. And similarly it can be proved that $\boldsymbol{S}_{n+1}\vdash A^\varphi\supset R'_n$. And by using the lemma 6, $\boldsymbol{LC}\vdash A^\varphi\supset R'_n$.

Corollary 11. *If* $\boldsymbol{S}_n\supsetneqq\boldsymbol{L}\supset\boldsymbol{S}_{n+1}$, *then* $\boldsymbol{L}\supset\subset\boldsymbol{S}_{n+1}$.

Proof. Suppose that $\boldsymbol{S}_{n+1}\not\supset\boldsymbol{L}$. Then there is a formula A such that $\boldsymbol{L}\vdash A$ but not that $\boldsymbol{S}_{n+1}\vdash A$. Obviously $\boldsymbol{S}_n\vdash A$. Hence by the theorem, $\boldsymbol{LC}+A\supset\subset\boldsymbol{S}_n$. This implies that $\boldsymbol{L}\supset\boldsymbol{S}_n$. This is contrary to the hypothesis.

Further we have the following

Theorem 12. *If A is an* I *formula, the system* $\boldsymbol{LC}+A$ *of the theorem* 4 *is separable.*

Proof. As is proved in [2], $\boldsymbol{LC}+P_n$ is a separable axiomatization for $\boldsymbol{S}_n$. Let A^φ be constructed as in the second proof of the theorem 4. Let B be the formula obtained from A^φ by substituting a_i's by a_{n+2-i}. Since A is an I formula, B is also an I formula. And it is easily seen that $\boldsymbol{LC}\vdash B\supset P_n$. And by the separability of $\boldsymbol{LC}$, P_n has an I proof in $\boldsymbol{LC}+B$ and hence in $\boldsymbol{LC}+A$. So the system $\boldsymbol{LC}+A$ is separable.

An example of such A is the following formula,

$$Q_n=((a_0\supset a_1)\supset b)\supset((a_1\supset a_2)\supset b)\supset\cdots\supset((a_n\supset a_{n+1})\supset b)\supset b,$$

in which we associate to the right. It is easily seen that Q_n is interdeducible with R_n in $\boldsymbol{LI}$. Hence $\boldsymbol{LI}+Q_n$, instead of $\boldsymbol{LC}+Q_n$, is sufficient for the axiomatization of $\boldsymbol{S}_n$. These formulas R'_n and Q_n were suggested to the author while he was talking with S. Nagata. If we substitute a_{2m}'s by a_0 and a_{2m+1}'s by a_1 in Q_n, we obtain the formula Z. The theorem 4 has been proved independently by Nagata using the proof-theoretic method (unpablished).

References

[1] M. Hanazawa: A characterization of axiom schema playing the rôle of Tertium non Datur in intuitionistic logic. Proc. Japan Acad., **42**, 1007–1010 (1966).

[2] T. Hosoi: The separable axiomatization of the intermediate propositional systems S_n of Gödel. Proc. Japan Acad., **42**, 1001-1006 (1966).
[3] ——: On the axiomatic method and the algebraic method for dealing with propositional logics. (To appear).