82. A Unique Continuation Theorem for Solutions of the Schrödinger Equations

By Kyûya MASUDA
Department of Mathematics, University of Tokyo

(Comm. by Zyoiti SUETUNA, M.J.A., May 12, 1967)

1. Introduction. We are concerned with the unique continuation property of solutions of the equation

$$(1)\qquad i^{-1}\frac{\partial}{\partial t}u=\sum_{j=1}^{n}[i\,\partial/\partial x_j+b_j(x)]^2u+q(x)u$$

for t in $(-\infty, \infty)$, and x in R^n, where $b_j(x)$ are real-valued functions of class C^2. We assume that the real-valued measurable function $q(x)$ satisfies the conditions (a) $q(x)$ is square-integrable over any compact set in R^n, (b) $q\{x\}$ is locally ess. bounded in a connected open set R^n-Q, and Q is of measure zero, (c) the operator H is essentially self-adjoint in $\boldsymbol{L}^2=\boldsymbol{L}^2(R^n)$, where H is an operator defined by $\boldsymbol{D}(H)=C_0^\infty(R^n)$, and $H\varphi=\sum_{j=1}^{n}[i\partial/\partial x_j+b_j]^2\varphi+q\varphi$, $\boldsymbol{D}(H)$ being the domain of H, (d) the spectral set of the self-adjoint extension $\boldsymbol{H}$ of H is bounded from below.

Some equations, appearing in current structure of quantum mechanics, are of the form (1). However the Schrödinger equations with the Stark effect potential are excluded from our consideration, since the spectral set of their Hamiltonian operators is, generally, not bounded from below (see Kato [1, 2], Stummel [3], Wienholtz [4], Ikebe-Kato [5]). The purpose of the present note is to show the following

Theorem. *Let u be a weak solution of* (1). *If u vanishes in some nonempty open subset G of the (x, t)-space R^{n+1}, then u vanishes identically.*

Here by a *weak solution* of (1) we mean a function $u(\cdot, t)$ of t with the following properties:

(i) $u(\cdot, t)$ takes values in $\boldsymbol{L}^2$, and is strongly continuous for t in $(-\infty, \infty)$ with respect to the norm of $\boldsymbol{L}^2$; (ii) we have

$$\int_{-\infty}^{\infty}\int_{R^n}u(x, t)\{i^{-1}\partial/\partial_t\overline{\Phi(x, t)}+\sum_{j=1}^{n}[-i\,\partial/\partial x_j+b_j]^2\overline{\Phi(x, t)}+q(x)\overline{\Phi(x, t)}\}dx\,dt=0$$

for any Φ in $C_0^\infty(R^n\times(-\infty, \infty))$, where $\bar{b}$ is the complex conjugate of b.

2. Proof of the theorem. Let $u_0(x)=u(x, 0)$. We define

$$S(z)u_0=\int_m^\infty \exp(iz\lambda)dE(\lambda)u_0\ ,$$

where $E(\lambda)$ is the resolution of the identity for the operator $\boldsymbol{H}$, and $m = inf\ \sigma(\boldsymbol{H})$. Then we have

Lemma. *$S(t)u_0 = u(\cdot, t)$ for any real t.*

Proof. As is easily seen, $S(t)u_0$ is a weak solution of (1) with the initial value $S(0)u_0 = u_0$. Let $w(\cdot, t) = S(t)u_0 - u(\cdot, t)$. Let us mollify $w(x, t)$ in t: $w_\varepsilon(x, t) = \int_{-\infty}^{\infty} \rho_\varepsilon(t-s)w(x, s)ds$, $\rho_\varepsilon(t)$ being the Friedrichs mollifier. For any φ in $C_0^\infty(R^n)$ and any real t, we have

$$\frac{d}{ds}(w_\varepsilon(\cdot, s), S(s-t)\varphi) = (w_\varepsilon(\cdot, s), -i\boldsymbol{H}S(s-t)\varphi) + (w_\varepsilon(\cdot, s), i\boldsymbol{H}s(t-s)\varphi) = 0,$$

$(\cdot, \cdot)$ being the scalar product of $\boldsymbol{L}^2$. Hence, $(w_\varepsilon(\cdot, t), \varphi) = (w_\varepsilon(\cdot, 0), S(-t)\varphi)$. Letting $\varepsilon \downarrow 0$, we have $(w(\cdot, t), \varphi) = 0$ for any real t and any φ in $C_0^\infty(R^n)$, showing that for each t, w equals zero as an element of $\boldsymbol{L}^2$.

Proof of the theorem. Let $U \times (t_1, t_2)$ be a nonempty open set contained in G. Then we have, by the Lemma,

$$(S(t)u_0, \varphi) = 0 \tag{2}$$

for any φ in $C_0^\infty(U)$ and any t in (t_1, t_2). Since $(S(z)u_0, \varphi)$ is holomorphic in the upper half-plane $\{z; \operatorname{Im} z > 0\}$, and is continuous in z with $\operatorname{Im} z \geq 0$, it follows from (2), and Schwarz's reflexion principle that $(S(z)u_0, \varphi)$ can be extended holomorphically into the lower half-plane across the interval (t_1, t_2). Thus, applying the unique continuation theorem in complex analysis, we have, by (2), $(S(z)u_0, \varphi) = 0$ for any φ in $C_0^\infty(U)$ and any z with $\operatorname{Im} z \geq 0$. Setting $v(x, \xi, \eta) = S(\xi + i\eta)u_0(x)$, we have $v(x, \xi, \eta) = 0$ for almost everywhere (x, ξ, η) in $U \times R_+^2$, where $R_+^2 = \{(\xi, \eta); -\infty < \xi < \infty, \eta > 0\}$. Hence we have

$$v(x, \xi, \eta) = 0 \tag{3}$$

for almost everywhere (x, ξ, η) in $(U \cap (R^n - Q)) \times R_+^2$. Here we note that $v(x, \xi, \eta)$ is square-integrable over any compact set in $R^n \times R_+^2$, since $(S(z)u_0, S(z)u_0)$ is continuous in z with $\operatorname{Im} z \geq 0$.

We shall show that $v(x, \xi, \eta)$ satisfies the equation

$$\left(v, \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} - \sum_{j=1}^n \left[i\frac{\partial}{\partial x_j} + b_j \right]^2 \Phi + \frac{\partial \Phi}{\partial \eta} - q\Phi \right)_{L^2(\boldsymbol{Z})} = 0 \tag{4}$$

for any Φ $C_0^\infty(\boldsymbol{Z})$, where $\boldsymbol{Z} = R^n \times R_+^2$, and $(\cdot, \cdot)_{L^2(\boldsymbol{Z})}$ is the scalar product of $L^2(\boldsymbol{Z})$. For any φ in $C_0^\infty(R^n)$, $(v(\cdot, \xi, \eta), \varphi)$ is a harmonic function of ξ and η, since $(S(z)u_0, \varphi)$ is holomorphic in the upper half-plane. Hence we have

$$(v, (\partial^2/\partial \xi^2 + \partial^2/\partial \eta^2)\varphi\psi)_{L^2(\boldsymbol{Z})} = 0 \tag{5}$$

for any φ in $C_0^\infty(R^n)$ and any ψ in $C_0^\infty(R_+^2)$. On the other hand, we have

$$\begin{aligned} &(v, -\partial/\partial\eta(\psi\varphi) + \textstyle\sum_{j=1}^n [i\partial/\partial x_j + b_j]^2 \psi\varphi + q\psi\varphi)_{L^2(\boldsymbol{Z})} \\ &= -\int_{R_+^2} \partial/\partial\eta \psi(\xi, \eta) d\xi d\eta \int_m^\infty \exp(i\xi\lambda - \eta\lambda) d(E(\lambda)u_0, \varphi) \\ &\quad + \int_{R_+^2} \psi(\xi, \eta) d\xi d\eta \int_m^\infty \lambda \exp(i\xi\lambda - \eta\lambda) d(E(\lambda)u_0, \varphi) = 0 \end{aligned} \tag{6}$$

for any φ in $C_0^\infty(R^n)$ and any ψ in $C_0^\infty(R_+^2)$. Hence, by (5) and (6), we see that *Eq.* (4) holds for any $\Phi(x, \xi, \eta)=\varphi(x)\psi(\xi, \eta)$ with φ in $C_0^\infty(R^n)$, and ψ in $C_0^\infty(R_+^2)$. Since the totality of finite sums $\sum_j \varphi_j\psi_j$ with $\varphi_j \in C_0^\infty(R^n)$ and $\psi_j \in C_0^\infty(R_+^2)$ is dense in $C_0^\infty(\boldsymbol{Z})$ in the topology of $\Phi(\boldsymbol{Z})$ (see L. Schwartz [6] p. 107), we see that Eq. (4) holds for any Φ in $C_0^\infty(\boldsymbol{Z})$. Since $v(x, \xi, \eta)$ is square-integrable over any compact set in $\boldsymbol{Z}$, applying the theorem on the interior regularity of weak solutions for elliptic equation (see F. Browder [7] p. 129), we see that $v(x, \xi, \eta)$ has strongly continuous $\boldsymbol{L}^2$-derivatives up to order 2 on any compact set in $(R^n-Q)\times R_+^2$, and satisfies the equation

$$(\partial^2/\partial\xi^2+\partial^2/\partial\eta^2)v-\sum_{j=1}^n[i\,\partial/\partial x_j+b_j]^2v-\partial/\partial\eta v-qv=0$$

in the generalized sense. Hence, applying the unique continuation theorem for solutions of elliptic equations of second order (see Aronszajn [8]), we have, by (3), $v(x, \xi, \eta)=0$ for almost everywhere (x, ξ, η) in $(R^n-Q)\times R_+^2$, since $(R^n-Q)\times R_+^2$ is connected in $\boldsymbol{Z}$ and $(U\cap(R^n-Q))\times R_+^2$ is a non-empty open set in $(R^n-Q)\times R_+^2$ by Assumption (b). Noting that for any φ in $C_0^\infty(R^n)$, $(v(x, \xi, \eta), \varphi)$ is continuous for (ξ, η) in $(-\infty, \infty)\times[0, \infty)$, we have

$$(v, \varphi)=0, \ -\infty<\xi<\infty, \ \eta\geq 0$$

for any φ in $C_0^\infty(R^n-Q)$. Hence, $(S(\xi)u_0, \varphi)=(v(\cdot, \xi, 0), \varphi)=0$. Since Q is of measure zero by Assumption (b), it follows from the above equality and the Lemma that for any $t, u(\cdot, t)$ equals zero as an element of $\boldsymbol{L}^2$. Thus the theorem is proved.

Remark. Combining the above theorem with the result recently obtained by the author [9], we have the following result. Let u_1 be a weak solution of (1), and u_0 be a weak solution of the equation $i^{-1}\partial u_0/\partial t=\sum_{j=1}^n[i\partial/\partial x_j+b_j]^2u_0$, $-\infty<t<\infty, x\in R^n$. If there exists a nonempty open set U with $U\subset$ the support of q such that $\|u_1(\cdot, t)-u_0(\cdot, t)\|_{L^2(U)}\leq c\exp(-\varepsilon t)$ for any $t\geq 0$, where c and ε are positive constants in t, then for any t, both $u_0(\cdot, t)$ and $u_1(\cdot, t)$ equal zero as an element of $\boldsymbol{L}^2$.

References

[1] Kato, T.: Fundamental properties of Hamiltonian operators of Schrödinger type. Trans. Amer. Math. Soc., **70**, 195-211 (1951).

[2] ——: On the existence of solutions of the helium wave equation. Trans. Amer. Math. Soc., **70**, 212-218 (1951).

[3] Stummel, F.: Singuläre elliptische Differentialoperatoren in Hilbertschen Räumen. Math. Ann., **132**, 150-176 (1956).

[4] Wienholtz, E.: Halbbeschränkte partielle Differentialoperatoren zweiter Ordnung vom elliptischen typus. Math. Ann., **135**, 50-80 (1958).

[5] Ikebe, T., and Kato, T.: Uniqueness of the self-adjoint extension of singular elliptic differential operators. Arch. Rat. Mech. Analy., **12**, 77-92 (1961).

[6] Schwartz, L.: Théorie des Distributions, Vol. 1. Hermann, Paris (1950).

[7] Browder, F.: Functional analysis and partial differential equations. II. Math. Ann., **145**, 81-226 (1962).
[8] Aronszajn, N.: A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. J. de Math., **36**, 235-247 (1957).
[9] Masuda, K.: Asymptotic behavior in time of solutions for evolution equations. To appear in the Journal of Functional Analysis.