# 103. On Maharam Subfactors of Finite Factors

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1. H. A. Dye [2] has laboriously investigated the structure of measure preserving transformations. In his study, Maharam's lemma plays an eminent role.

It seems natural to consider that a non-commutative version of Maharam's lemma is useful in the theory of von Neumann algebras. We shall introduce a notion of Maharam subalgebra (cf. Definition in § 2), motivated by Maharam's lemma.

In this paper, we shall treat subfactors of $II_1$-factors which are Maharam subalgebras. Maharam subalgebras in general von Neumann algebras of finite type will be discussed in a subsequent paper.

2. In the first place, we shall state briefly main properties of the conditional expectation of a finite von Neumann algebra introduced and discussed by H. Umegaki [5].

Let $\mathcal{A}$ be a finite factor, then there exists a unique faithful normal trace $\phi$ on $\mathcal{A}$ such that $\phi(I)=1$. Let $\mathcal{B}$ be a subfactor of $\mathcal{A}$. Then for each $A$ in $\mathcal{A}$, there exists a normal linear mapping $A\rightarrow A^{\varepsilon}$ of $\mathcal{A}$ onto $\mathcal{B}$ which has the following properties:

(1) $\phi(AB)=\phi(A^{\varepsilon}B)$, for $A\in\mathcal{A}$ and $B\in\mathcal{B}$,

(2) $A^{\varepsilon}=0$ and $A\geqq 0$ implies $A=0$,

(3) $A\geqq 0$ implies $A^{\varepsilon}\geqq 0$,

(4) $A^{*\varepsilon}=A^{\varepsilon *}$,

(5) $(AB)^{\varepsilon}=A^{\varepsilon}B$, for $A\in\mathcal{A}$ and $B\in\mathcal{B}$,

(6) $I^{\varepsilon}=I$,

(7) $(AB)^{\varepsilon}=(BA)^{\varepsilon}$, for $A\in\mathcal{A}$ and $B\in\mathcal{A}\cap\mathcal{B}'$.

The mapping $\varepsilon$ will be called the *conditional expectation* of $\mathcal{A}$ relative to $\mathcal{B}$. The conditional expectation is uniquely determined by (1).

Now, we shall introduce the following

**Definition.** Let $\mathcal{A}$ be a finite factor, $\mathcal{B}$ a subfactor of $\mathcal{A}$ and $\varepsilon$ the conditional expectation of $\mathcal{A}$ relative to $\mathcal{B}$. Then $\mathcal{B}$ is called a *Maharam subalgebra* of $\mathcal{A}$ if for any $A$ in $\mathcal{B}$ such that $0\leqq A\leqq 1$, there exists a projection $E$ in $\mathcal{A}$ such that

$$E^{\varepsilon}=A.$$

The following properties on Maharam subalgebras are clear by the definition: Let $\mathcal{A}$ be a finite factor, $\mathcal{B}$ a subfactor of $\mathcal{A}$ and $\mathcal{C}$ a subfactor of $\mathcal{B}$, then

(i) If $\mathcal{B}$ is a Maharam subalgebra of $\mathcal{A}$, then $\mathcal{C}$ is a Maharam subalgebra of $\mathcal{A}$,

and

(ii) if $\mathcal{C}$ is a Maharam subalgebra of $\mathcal{B}$, then $\mathcal{C}$ is a Maharam subalgebra of $\mathcal{A}$.

These properties seem to indicate that the distance between the whole algebra and the Maharam subalgebras is sufficiently large.

A sufficient condition that a subfactor of a finite factor is a Maharam subfactor will be given in the following

**Theorem 1.** *Let $\mathcal{A}$ be a finite factor and $\mathcal{B}$ be a subfactor of $\mathcal{A}$. If the relative commutant $\mathcal{B}^c=\mathcal{A}\cap\mathcal{B}'$ is a $II_1$-factor, then $\mathcal{B}$ is a Maharam subalgebra of $\mathcal{A}$.*

**Proof.** Let $A$ be an arbitrary operator of $\mathcal{B}$ such that $0\leqq A\leqq I$. Then there exists a resolution of the identity $E_\lambda$, the one-parameter family of projections in $\mathcal{B}$ such that

$$A=\int_0^1 \lambda dE_\lambda.$$

Let us put

$$F(n, i)=E_{i2^{-n}}-E_{(i-1)2^{-n}}$$

and

$$A_n=\sum_{i=1}^{2^n}(i-1)2^{-n}F(n, i),$$

for $n=1, 2, \cdots$ and $i=1, 2, \cdots, 2^n$. Then $A_n$ converges strongly to $A$.

Let $\epsilon$ be the conditional expectation of $\mathcal{A}$ relative to $\mathcal{B}$. Because $\mathcal{A}\cap\mathcal{B}'$ is a $II_1$-factor, there exists a family

$$\{E(n, i); n=1, 2, \cdots, i=1, 2, \cdots, 2^n\}$$

of projections in $\mathcal{A}\cap\mathcal{B}'$ which satisfies

$$E(n, i)^\epsilon=(i-1)2^{-n}, \tag{8}$$

$$E(n+1, 2i-1)=E(n, i), \tag{9}$$

and

$$E(n+1, 2i)\geqq E(n, i), \tag{10}$$

for every $n$ and $i$. Define

$$G_n=\sum_{i=1}^{2^n}E(n, i)F(n, i).$$

Then $G_n$ is a projection of $\mathcal{A}$ since $E(n, i)\in\mathcal{A}\cap\mathcal{B}'$, $F(n, i)\in\mathcal{B}$ and

$$F(n, i)F(n, j)=0 \qquad \text{for } i\neq j.$$

By (9), (10) and

$$F(n, i)=F(n+1, 2i-1)+F(n+1, 2i),$$

we have

$$G_n \leqq G_{n+1} \quad \text{for } n=1, 2, \cdots$$

Hence $G_n$ converges strongly to a projection $G$ in $\mathcal{A}$. By the linearity of the conditional expectation $\varepsilon$, (5) and (8), we have

$$G_n^{\varepsilon} = \sum_{i=1}^{2^n} E(n, i)^{\varepsilon} F(n, i)$$
$$= \sum_{i=1}^{2^n} (i-1)2^{-n} F(n, i) = A_n.$$

On the other hand, $\varepsilon$ is strongly continuous in the unit sphere of $\mathcal{A}$. Therefore,

$$G^{\varepsilon} = s\text{-}\lim_{n\to\infty} G_n^{\varepsilon} = s\text{-}\lim_{n\to\infty} A_n = A,$$

that is, $\mathcal{B}$ is a Maharam subalgera of $\mathcal{A}$.

**3.** We shall begin with a proof of the following lemma which may be well-known among specialists:

**Lemma 1.** *Let $\mathcal{A}$ and $\mathcal{A}_1$ (resp. $\mathcal{B}$ and $\mathcal{B}_1$) be semi-finite von Neumann algebras acting on a Hilbert space $\mathfrak{H}$ (resp. $\mathfrak{K}$), then*

$$(\mathcal{A}\otimes\mathcal{B}) \cap (\mathcal{A}_1\otimes\mathcal{B}_1) = (\mathcal{A}\cap\mathcal{A}_1)\otimes(\mathcal{B}\cap\mathcal{B}_1),$$

*on $\mathfrak{H}\otimes\mathfrak{K}$.*

**Proof.** First we shall show for $\mathcal{B}=\mathcal{B}_1=\mathcal{L}(\mathfrak{K})$. Since

$$[(\mathcal{A}\otimes\mathcal{L}(\mathfrak{K}))\cap(\mathcal{A}_1\otimes\mathcal{L}(\mathfrak{K}))]' = \boldsymbol{R}(\mathcal{A}'\otimes \boldsymbol{C}_{\mathfrak{K}}, \mathcal{A}_1'\otimes\boldsymbol{C}_{\mathfrak{K}})$$
$$= \boldsymbol{R}(\mathcal{A}', \mathcal{A}_1')\otimes\boldsymbol{C}_{\mathfrak{K}}$$

and

$$[(\mathcal{A}\cap\mathcal{A}_1)\otimes\mathcal{L}(\mathfrak{K})]' = (\mathcal{A}\cap\mathcal{A}_1)'\otimes\mathcal{L}(\mathfrak{K})'$$
$$= \boldsymbol{R}(\mathcal{A}', \mathcal{A}_1')\otimes\boldsymbol{C}_{\mathfrak{K}},$$

we have

(11) $$\mathcal{A}\otimes\mathcal{L}(\mathfrak{K})\cap\mathcal{A}_1\otimes\mathcal{L}(\mathfrak{K}) = (\mathcal{A}\cap\mathcal{A}_1)\otimes\mathcal{L}(\mathfrak{K}).$$

Similarly, we have

(12) $$\mathcal{L}(\mathfrak{H})\otimes\mathcal{B}\cap\mathcal{L}(\mathfrak{H})\otimes\mathcal{B}_1 = \mathcal{L}(\mathfrak{H})\otimes(\mathcal{B}\cap\mathcal{B}_1).$$

Since $\mathcal{A}, \mathcal{A}_1, \mathcal{B}$, and $\mathcal{B}_1$ are semi-finite, by [1; p. 30] we have

(13) $$\mathcal{A}\otimes\mathcal{B} = \mathcal{A}\otimes\mathcal{L}(\mathfrak{K})\cap\mathcal{L}(\mathfrak{H})\otimes\mathcal{B},$$

and

(14) $$\mathcal{A}_1\otimes\mathcal{B}_1 = \mathcal{A}_1\otimes\mathcal{L}(\mathfrak{K})\cap\mathcal{L}(\mathfrak{H})\otimes\mathcal{B}_1.$$

These equalities (11)-(14) together imply

$$(\mathcal{A}\otimes\mathcal{B})\cap(\mathcal{A}_1\otimes\mathcal{B}_1)$$
$$= \mathcal{A}\otimes\mathcal{L}(\mathfrak{K})\cap\mathcal{L}(\mathfrak{H})\otimes\mathcal{B}\cap\mathcal{A}_1\otimes\mathcal{L}(\mathfrak{K})\cap\mathcal{L}(\mathfrak{H})\otimes\mathcal{B}_1$$
$$= (\mathcal{A}\cap\mathcal{A}_1)\otimes\mathcal{L}(\mathfrak{K})\cap\mathcal{L}(\mathfrak{H})\otimes(\mathcal{B}\cap\mathcal{B}_1)$$
$$= (\mathcal{A}\cap\mathcal{A}_1)\otimes(\mathcal{B}\cap\mathcal{B}_1),$$

which is the desired.

The following lemma is due to Powers [3; Lemma 3.3]. For the sake of completeness, we shall list here a proof.

**Lemma 2.** *Let $\mathcal{A}$ be a finite factor, and $\mathcal{B}$ and $\mathcal{C}$ $I_n$-subfactors of $\mathcal{A}$, then there exists a unitary operator $U$ in $\mathcal{A}$ such that $U\mathcal{B}U^* = \mathcal{C}$.*

**Proof.** Let $\phi$ be the normalized trace of $\mathcal{A}$. Since $\mathcal{B}$ (resp. $\mathcal{C}$) is $I_n$-factor, there exists a system of matrix units

$$\{W_{ij};\ i,j=1,2,\cdots,n\}\ (\text{resp.}\{V_{ij};\ i,j=1,2,\cdots,n\})$$

which spans $\mathcal{B}$ (resp. $\mathcal{C}$). Since

$$I=\sum_{i=1}^{n}W_{ii} \quad \text{and} \quad I=\sum_{i=1}^{n}V_{ii},$$

we have

$$1=\sum_{i=1}^{n}\phi(W_{ii}) \quad \text{and} \quad 1=\sum_{i=1}^{n}\phi(V_{ii}). \tag{15}$$

Since the matrix units satisfy

$$W_{ii}=W_{1i}{}^{*}W_{1i} \quad \text{and} \quad W_{11}=W_{1i}W_{1i}{}^{*}$$
$$(\text{resp. } V_{ii}=V_{1i}{}^{*}V_{1i} \quad \text{and} \quad V_{11}=V_{1i}V_{1i}{}^{*}),$$

$W_{ii}$ (resp. $V_{ii}$) is equivalent to $W_{11}$ (resp. $V_{11}$) with respect to $\mathcal{A}$ Therefore, by (15), we have

$$\phi(W_{11})=\frac{1}{n}=\phi(V_{11}).$$

This implies that $W_{11}$ is equivalent to $V_{11}$, and that there is a partially isometric operator $V$ in $\mathcal{A}$ such that

$$W_{11}=V^{*}V \quad \text{and} \quad V_{11}=VV^{*}.$$

Let us define

$$U=\sum_{i=1}^{n}V_{i1}VW_{1i}.$$

Then it is easy to compute that $U$ is a unitary operator in $\mathcal{A}$ and

$$UW_{ij}U^{*}=V_{ij} \qquad \text{for } i,j=1,2,\cdots,n.$$

This proves the lemma.

**Lemma 3.** *Let $\mathcal{A}$ and $\mathcal{B}$ be isomorphic finite factors. If $\mathcal{A}_1$ (resp. $\mathcal{B}_1$) is a $I_n$-subfactor of $\mathcal{A}$ (resp. $\mathcal{B}$), then the relative commutant $\mathcal{A}_1^c=\mathcal{A}\cap\mathcal{A}_1'$ is isomorphic to $\mathcal{B}_1^c=\mathcal{B}\cap\mathcal{B}_1'$.*

**Proof.** Let $\Phi$ be the isomorphism of $\mathcal{A}$ onto $\mathcal{B}$. Write $\mathcal{B}_2=\Phi(\mathcal{A}_1)$. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are $I_n$-subfactors of $\mathcal{B}$. By Lemma 2, there exists a unitary operator $U$ in $\mathcal{B}$ such that $U\mathcal{B}_1U^{*}=\mathcal{B}_2$. Hence $\mathcal{B}_1^c$ is isomorphic to $\mathcal{B}_2^c$. On the other hand, $\mathcal{A}_1^c$ is isomorphic to $\mathcal{B}_2^c$. Therefore $\mathcal{A}_1^c$ is isomorphic to $\mathcal{B}_1^c$.

The following theorem may shed light on the notion of Maharam subalgebras:

**Theorem 2.** *Let $\mathcal{A}$ be a $II_1$-factor acting on a Hilbert space $\mathfrak{H}$ and $\mathcal{B}$ a $I_n$-subfactor of $\mathcal{A}$, then $\mathcal{B}$ is a Maharam subfactor of $\mathcal{A}$.*

**Proof.** By (ii), it is sufficient to show that $\mathcal{B}$ is a Maharam subfactor of the hyperfinite factor since a $I_n$-subfactor in a $II_1$-factor is contained in a hyperfinite subfactor. Hence it is sufficient to assume that $\mathcal{A}$ itself is hyperfinite. Then $\mathcal{A}$ is isomorphic to $\mathcal{A}\otimes\mathcal{B}$. The relative commutant of $C_{\mathfrak{H}}\otimes\mathcal{B}$ in $\mathcal{A}\otimes\mathcal{B}$ is $\mathcal{A}\otimes C_{\mathfrak{H}}$ by

Lemma 1, and $\mathscr{B}^c=\mathscr{B}'\cap\mathscr{A}$ is a $II_1$-factor by Lemma 3. Now, the theorem is a consequence of Theorem 1.

Without appealing the hyperfinite subfactor, the another proof of the theorem is also possible based on [4; Lemma 4].

The class $\mathfrak{M}$ of all Maharam subfactors of a $II_1$-factor contains all subfactors of type I by Theorem 2. However, the class $\mathscr{M}$ is wider than the class of all of all subfactors of type I. This is a consequence of the following

**Theorem 3.** *In a $II_1$-factor $\mathscr{A}$, there exists a Maharam $II_1$-subfactor $\mathscr{B}$ of $\mathscr{A}$.*

**Proof.** By (ii), we can assume, as in the proof of Theorem 2, $\mathscr{A}$ is hyperfinite. It is well-known that $\mathscr{A}$ is isomorphic to $\mathscr{A}\otimes\mathscr{A}$. Applying Lemma 1, we have a $II_1$-subfactor $\mathscr{B}$ of $\mathscr{A}$ such that $\mathscr{A}\cap\mathscr{B}'$ is a $II_1$-factor. By Theorem 1, $\mathscr{B}$ is a Maharam subfactor which is also of type II.

The existence of a non Maharam proper subfactor of a certain $II_1$-factor will be discussed on another occasion.

## References


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