97. The Asymptotic Formula for the Trace of Green Operators of Elliptic Operators on Compact Manifold*)

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§ 1. **Preliminaries.** It is interesting to know the asymptotic expansion of the trace of the Green operator $(P+\tau)^{-1}$ of elliptic pseudo-defferential operator P operating on sections of a complex vector bundle X over a compact differentiable manifold M. The asymptotic expansion is obtained by introducing a special class of pseudo-differential operators on $X\otimes 1_{R^1}$, where 1_{R^1} is the trivial line bundle over the real line $\boldsymbol{R}^1$. This is called β-pseudo-differential operator for the time being. Proofs are omitted, but will be published elsewhere.

In the following, we shall follow the usual notations in [1] or [2] for the special spaces of distributions.

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§ 2. **β-pseudo-differential operators.** Consider a σ-compact differentiable n-manifold M and a smooth complex vector bundle X of dimension l over M. Let Y be the bundle over $M\times\boldsymbol{R}^1$ induced from X by the projection $M\times\boldsymbol{R}^1\to M$. We identify the bundle Y with $X\otimes 1_{R^1}$. We denote the generic point of $M\times\boldsymbol{R}^1$ by (x, s). When Z is a vector bundle over a differentiable manifold N, we denote the space of C^∞ sections (resp. C^∞ sections with compact support) over an open subset U of N by $\mathcal{E}(U, Z)$ (resp. $\mathcal{D}(U, Z)$). In the following, we denote the annulus $\{(\rho, \sigma)\in\boldsymbol{R}^2 : \frac{1}{2}\le\rho^2+\sigma^2\le 2\}$ by A.

Definition 1. A continuous linear map P from $\mathcal{D}(M, X)\widehat{\otimes}\mathcal{S}'(\boldsymbol{R}^1)$ into $\mathcal{E}(M, X)\widehat{\otimes}\mathcal{S}'(R^1)$ is called a β-pseudo-differential operator of order s_0, if there is a sequence $s_0>s_1>\cdots\to-\infty$ of reals such that, for all $f\in\mathcal{D}(M, X)$ and $g\in\mathcal{E}(M)$ which is real valued with $dg\neq 0$ on supp f, $e^{-i\lambda(g\rho+s\sigma)}P(fe^{i\lambda(g\rho+s\sigma)})$ is the pull back of a section $p(f, g\rho, x, \sigma)$ of X and there holds the asymptotic expansion

$$p(f, g\rho, x, \sigma)\sim\sum_0^\infty p_j(f, g\rho, x, \sigma)\lambda^{s_j}, \qquad \lambda\to\infty, \tag{2.1}$$

which has the following property: For any integer $N>0$ and com-

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pact set $\mathcal{K}$ of real functions g in $\mathcal{E}(M)$ with $dg \neq 0$ on supp f,

(2.2) $$\lambda^{-s_N}\left(p(f, g\rho, x, \sigma) - \sum_0^{N-1} p_j(f, g\rho, x, \sigma)\lambda^{s_j}\right)$$

remains bounded in $\mathcal{E}(M\times A, X\otimes 1_A)$ when $\lambda \to +\infty$. Here we denoted by 1_A the trivial line bundle over A.

Since this is a slight modification of the Definition 2.1 in Hörmander [3], almost all properties stated in [3] hold with obvious modifications. Especially, the generalized Leibniz formula for composition of two β-pseudo-differential operators is valid in our case. Therefore, if Q is elliptic, we can easily find another β-pseudo-differential operator R such that with some β-pseudo-differential operators $Q_{-\infty}$, $Q'_{-\infty}$ of order $-\infty$,

(2.3) $$R\circ Q = I + Q_{-\infty},\ Q\circ R = I + Q'_{-\infty}.$$

§ 3. The asymptotic formula for green kernels. Throughout this section we assume that M is compact and that P is an elliptic pseudo-differential operator of order $2m$ in $\mathcal{D}(M, X)$. Further, we assume that a measure $d\mu$ on M and hermitian metric $(\ |\ )$ on X are given and that the principal symbol $P_0(x, \xi)$, $x\in M$ and $\xi \in T_x^*(M)$, of P satisfies arg $(P_0(x, \xi)u \mid u) \neq \pm\pi$, $u \in X_x$.

Thus $Q = P + D_s^{2m}$, $s \in \boldsymbol{R}^1$, is also an elliptic β-pseudo-differential operator of order $2m$ on $\mathcal{D}(M, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^1)$.

Theorem 1. If τ_0 is sufficiently large, there is a β-pseudo-differential operator G operating on $\mathcal{D}(M, X)\hat{\otimes}\mathcal{S}'(\boldsymbol{R}^1)$ and satisfying

(3.1) $$(Q+\tau_0)G = I,\ G(Q+\tau_0) = I.$$

For any section $u \in \mathcal{D}(M, X)$ there is a section $v_\tau \in \mathcal{D}(M, X)$, $\tau \in \boldsymbol{R}^1$ such that the pull back of v_τ is equal to $e^{-is\tau}G(e^{is\tau}u)$. We denote the mapping $u \to v_\tau$ by E_τ.

Theorem 2. The mapping E_τ is a pseudo-differential operator, in the sense of Hörmander [3], of order $-2m$ operating in $\mathcal{D}(M, X)$ and satisfying

(3.2) $$(P+\tau^{2m}+\tau_0)E_\tau = I, \qquad E_\tau(p+\tau^{2m}+\tau_0) = I.$$

If $2m \geq n+1$, the kernel of the operator $E_\tau = (P+\tau^{2m}+\tau_0)^{-1}$ is continuous. For any point x (resp. y) in M, let φ_1 (resp. φ_2) be a function in $\mathcal{D}(M)$ such that $\varphi_1 \equiv 1$ in some neighbourhood of x (resp. $\varphi_2 \equiv 1$ in some neighbourhood of y). Further, we assume that the support of φ_1 and φ_2 are so small that we can trivialize the bundle X in some neighbourhood U of (supp φ_1) $\cup$ (supp φ_2). Thus operators are decomposed into $l\times l$ components. For any linear function $x\cdot\xi = x_1\xi_1 + \cdots + x_n\xi_n$ of coordinates $(x_1, \cdots, x_n)$ in U, set

$$g_{jk}(x, \xi, \tau) = e^{-i(x\cdot\xi + s\cdot\tau)}\varphi_1 G_{jk}(\varphi_2 e^{i(x\cdot\xi+s\sigma)}),$$

where G_{jk} is the j, k component of G in $U\times\boldsymbol{R}^1$. Then we have

Theorem 3.

$$E_{jk}(x, y, \tau) = (2\pi)^{-n}\rho(y)^{-1}\int_{\mathbf{R}^n} g_{jk}(x, \xi, \tau)e^{i(x-y)\cdot\xi}d\xi, \tag{3.3}$$

where $\rho(y)$ is the density of the measure $d\mu$ with respect to the measure $dx_1dx_2\cdots dx_n$.

Set $x=y$ and $\varphi_1=\varphi_2=\varphi$ in Theorem 3 and denote the asymptotic expansion of $e^{-i(x\cdot\xi+s\tau)}\varphi G_{jk}(\varphi e^{i(x\cdot\xi+s\tau)})$ by

$$e^{-i(x\cdot\xi+s\cdot\tau)}\varphi G_{jk}(\varphi e^{i(x\cdot\xi+s\cdot\tau)})\sim\sum_{r=0}^{\infty} g_{jk,r}(x, \xi, \tau),$$

where $g_{jk,r}(x, \xi, \tau)$, $r=0, 1, 2, \cdots$ are functions homogeneous in (ξ, τ) of degree $s_r\searrow-\infty$.

Theorem 4. The asymptotic expansion of trace E_τ is given by

$$\begin{aligned}\text{trace}\quad E_\tau &\sim\int_M \sum_j E_{jj}(x, x, \tau)\, d\mu(x)\\ &\sim(2\pi)^{-1}\sum_{k=0}^{\infty}\tau^{s_k+n}\int_M \frac{d\mu}{\rho(x)}\int_{R^n}\sum_j^l g_{jj}(x, \xi, 1)d\xi.\end{aligned} \tag{3.4}$$

Remark 1. $g_{jk,r}(x, \xi, \tau)$ are calculable through calculus of symbols.

Remark 2. After the author prepared this manuscript, he was communicated by Mr. K. Asano that Mr. Shimakura also obtained the formula (3.4) in different situation by a different method.

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