96. *Topology of Compact Sasakian Manifolds*

By Shûkichi TANNO

Mathematical Institute, Tôhoku University, Sendai

(Comm. by Zyoiti SUETUNA, M.J.A., June 12, 1967)

Introduction. The Betti numbers of compact Kählerian manifolds were studied by many geometers. Especially the second Betti numbers of compact Kählerian manifolds were studied by M. Berger [1], R. L. Bishop and S. I. Goldberg [2] and others. It is natural to research these problems in compact Sasakian manifolds, and in fact, they were studied by S.I. Goldberg [5], S. Tachibana [7], S. Tachibana and Y. Ogawa [8] and S. Tanno [10], etc.

1. In [10] we have used (m-1)-homothetic deformations to get results of the first Betti numbers. We call these deformations D-homothetic deformations, where D denotes the distribution defined by a contact form η. To get results of the second Betti numbers and harmonic forms, we also utilize a D-homothetic deformation:

$$g \rightarrow {}^*g = \alpha g + (\alpha^2 - \alpha)\, \eta \otimes \eta \tag{1.1}$$

of the associated Riemannian metric g for a positive constant α. Then if (ϕ, ξ, η, g) is a Sasakian structure for a contact form η, then $({}^*\phi = \phi, {}^*\xi = \alpha^{-1}\xi, {}^*\eta = \alpha\eta, {}^*g)$ is also a Sasakian structure. By studying the relations of harmonic forms with respect to g and ${}^*g$, we get

Theorem 1. *A compact m-dimensional Sasakian manifold M with sectional curvature* $> -3/(m\text{-}2)$ *has the first Betti number* $b_1(M) = 0$. *If* $m = 3$, *we have also* $b_2(M) = 0$.

A harmonic 2-form w is called of the hybrid type (pure type, resp.) if it satisfies

$$w(\phi X, \phi Y) = w(X, Y) \qquad (= -w(X, Y), \text{resp.}) \tag{1.2}$$

for any vector fields X and Y on M.

Theorem 2. *If* $m \geq 5$, *a compact Sasakian manifold M with sectional curvature* $> -3/(m\text{-}2)$ *has no harmonic* 2-*form of the pure type. And if sectional curvature* > 0, *there is no harmonic* 2-*form of the hybrid type. Especially, then, we have* $b_2(M) = 0$.

Remark. Under the additional condition that ξ is regular, S. I. Goldberg [5] obtained the last half of Theorem 2. In Kählerian case the similar fact was obtained by R. L. Bishop and S. I. Goldberg [2].

2. We denote by $K(X, Y)$ the sectional curvature for 2-plane determined by X and Y. As is well known in Kählerian manifolds holomorphic pinchings were studied by several authers. In Sasakian

manifolds, we want to define certain pinching for ϕ-holomorphic sectional curvatures. One of the conspicuous properties of Sasakian manifolds is that the sectional curvature for 2-plane which contains ξ is equal to 1. So when we normalize the metric we want to preserve this property, this is why we consider a D-homothetic deformation as a normalization. Namely we say that M is $^*\lambda$-holomorphically pinched if the following relation is satisfied

$$(2.2) \qquad {}^*\lambda \leq {}^*K(X, {}^*\phi X) \leq 1$$

for any X in D_x, $x \in M$.

Theorem 3. *If a compact Sasakian manifold M is $^*\lambda$-holomorphically pinched with $^*\lambda > -1$, then $b_2(M)=0$.*

Theorem 4. *If a Sasakian manifold is $^*\lambda$-holomorphically pinched with $^*\lambda > 1/4$, then it is of strictly positive curvature with respect to $^*g$. Further if $^*\lambda > -1/3$, then we have the second D-homothetic deformation $^*g \to {}^0g$ so that M is of strictly positive curvature with respect to ^0g.*

Theorem 5. *If a Sasakian manifold M is $^*\lambda$-holomorphically pinched with $^*\lambda > 17/35$, then M is Riemannian $^*\delta$-pinched with $^*\delta > 1/4$. Further if $^*\lambda > 1/5$, then by the second D-homothetic deformation $^*g \to {}^0g$, M is Riemannian $^0\delta$-pinched with $^0\delta > 1/4$. Thus if M is complete and simply connected M is homeomorphic to a sphere.*

3. From these theorems we can derive applications. For example, we have

Theorem 6. *If a complete, simply connected Sasakian manifold M is $^*\lambda$-holomorphically pinched with $^*\lambda > -1/3$ and if the scalar curvature is constant, then M is globally D-homothetic to the unit sphere.*

To prove this Theorem we need the recent results obtained by E.D. Moskal [6].

One of the other results is as follows:

Theorem 7. *Let M be a 5-dimensional, simply connected, compact Sasakian manifold with strictly positive curvature or $^*\lambda$-holomorphically pinched with $^*\lambda > -1$. If the torsion part of the integral second homology group of M vanishes, then M is diffeomorphic to a sphere.*

References

[1] M. Berger: Pincement riemannian et pincement holomorphe. Ann. Scuola Norm. Sup. Pisa, **14**, 151-159 (1960).

[2] R. L. Bishop and S. I. Goldberg: On the second cohomology group of a Kähler manifold of positive curvature. Proc. Amer. Math. Soc., **112**, 119-122 (1965).

[3] ——: On the topology of positively curved Kähler manifolds. Tôhoku Math. Jour., **15**, 359-364 (1963).
[4] ——: On the topology of positively curved Kähler manifolds. II. Tôhoku Math. Jour., **17**, 310-318 (1965).
[5] S. I. Goldberg: Rigidité de variété de contact à courbure positive. Compt. Rend. Acad. Sci. Paris, **261**, 1936-1939 (1965).
[6] E. D. Moskal: Contact manifolds of positive curvature. Thesis, Univ. Illinois (1966).
[7] S. Tachibana: On harmonic tensors in compact Sasakian spaces. Tôhoku Math. Jour., **17**, 271-284 (1965).
[8] S. Tachibana and Y. Ogawa: On the second Betti number of a compact Sasakian space. Nat. Sci. Rep. Ochanomizu Univ., **17**, 27-32 (1966).
[9] S. Tanno: Partially conformal transformations with respect to (m-1)-dimensional distributions of m-dimensional Riemannian manifolds. Tôhoku Math. Jour., **17**, 358-409 (1965).
[10] ——: Harmonic forms and Betti numbers of certain contact Riemannian manifolds. (To appear).
[11] ——: Sur une variété de K-contact métrique de dimension 3. Comp. Rend. Acad. Sci. Paris, **263**, 317-319 (1966).