94. On Integers Expressible as a Sum of Two Powers. II

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1. In a recent paper [2] we proved the following results:

Theorem 1. *There is n_0 such that for every $n \geqq n_0$ there are positive integers x and y satisfying*

$$n < x^h + y^h < n + cn^a,$$

where h is any integer $\geqq 2$,

$$a = \left(1 - \frac{1}{h}\right)^2 \quad and \quad c = h^{2-(1/h)}.$$

Theorem 2. *For any $\varepsilon > 0$, there is $n_0 = n_0(\varepsilon)$ such that for every $n \geqq n_0$ there are positive integers x and y satisfying*

$$n < x^f + y^h < n + (c+\varepsilon)n^a,$$

where f and h are any integers $\geqq 2$,

$$a = \left(1 - \frac{1}{f}\right)\left(1 - \frac{1}{h}\right) \quad and \quad c = hf^{1-(1/h)}.$$

The case $h=2$ of Theorem 1 and the case $f=h>2$ of Theorem 2 are due to Uchiyama [3], while the case $f=h=2$ of Theorem 2 is due to Bambah and Chowla [1].

As pointed out in Remark 4 of [2] we can replace c, in Theorem 2, by $C = fh^{1-(1/f)}$; but the theorem with c is the better result if $f > h$.

In this note we obtain the following refinement of Theorem 2 and generalization of Theorem 1:

Theorem 3. *There is n_0 such that for every $n \geqq n_0$ there are positive integers x and y satisfying*

$$n < x^f + y^h < n + cn^a,$$

where f and h are any integers such that $f \geqq h \geqq 2$,

$$a = \left(1 - \frac{1}{f}\right)\left(1 - \frac{1}{h}\right) \quad and \quad c = hf^{1-(1/h)}.$$

This follows from the case $h=2$ of Theorem 1 and

Lemma 1. *Theorem 3 is true for $f > 2$.*

The proof of this lemma has similarities with, but is more complicated than, the proofs of Theorems 1 and 2 and their special cases in [1], [2], and [3].

2. *Proof of Lemma 1.* We write $[t]$ for the greatest integer $\leqq t$.

Let b be a fixed constant such that

$$\frac{1}{2}f<b<f.$$

Suppose first that

(1) $$m=[n^{1/f}]\geqq(n-bn^{1-(1/f)})^{1/f}.$$

Then

$$\begin{aligned} n&<m^f+[(n-m^f)^{1/h}+1]^h \\ &\leqq m^f+((n-m^f)^{1/h}+1)^h \\ &\leqq n+h(bn^{1-(1/f)})^{1-(1/h)}(1+o(1)) \\ &<n+cn^a \end{aligned}$$

for large n, since $b<f$ and so

$$c=hf^{1-(1/h)}>hb^{1-(1/h)}.$$

Hence the lemma follows if (1) be true. We therefore assume in the rest of the proof that (1) is false; i.e., that

(2) $$m=[n^{1/f}]<(n-bn^{1-(1/f)})^{1/f}=M,$$

say.

Lemma 2. *Let* $f\geqq 3$,

$$N=(M+1)^f-M^f+1$$

and

$$g(n)=N-(N^{1/h}-1)^h.$$

Then $g(n)<cn^a$, *for large* n.

Proof. For large n,

$$\begin{aligned} M&=(n-bn^{1-(1/f)})^{1/f} \\ &=n^{1/f}\left(1-\frac{b}{f}n^{-1/f}+o(n^{-1/f})\right), \end{aligned}$$

$$\begin{aligned} N&=fM^{f-1}\left(1+\frac{1}{2}(f-1)M^{-1}+o(M^{-1})\right) \\ &=fn^{1-(1/f)}\left(1-b\left(1-\frac{1}{f}\right)n^{-1/f}+o(n^{-1/f})\right)\left(1+\frac{1}{2}(f-1)n^{-1/f}+o(n^{-1/f})\right) \\ &=fn^{1-(1/f)}\left(1-\left(b-\frac{1}{2}f\right)\left(1-\frac{1}{f}\right)n^{-1/f}+o(n^{-1/f})\right) \end{aligned}$$

and

$$\begin{aligned} g(n)&=hN^{1-(1/h)}\left(1-\frac{1}{2}(h-1)N^{-1/h}+o(N^{-1/h})\right) \\ &<h(fn^{1-(1/f)})^{1-(1/h)}=cn^a, \end{aligned}$$

since $b>\frac{1}{2}f$.

Suppose now that

$$(m+1)^f+1\leqq n+g(n).$$

Then Lemma 1 is clearly true. We therefore assume in the rest of the proof that

(3) $$(m+1)^f+1>n+g(n).$$

Since

$$n<m^f+[(n-m^f)^{1/h}+1]^h \leqq m^f+((n-m^f)^{1/h}+1)^h,$$

Lemma 1 now follows from Lemma 2 and

Lemma 3. (2) *and* (3) *imply that*

$$m^f+((n-m^f)^{1/h}+1)^h<n+g(n).$$

Proof. From (3),

$$n-m^f<(m+1)^f-m^f+1-g(n).$$

Clearly $(m+1)^f-m^f$ is a strictly increasing function of m. Hence, from (2),

$$\begin{aligned} n-m^f &<(M+1)^f-M^f+1-g(n) \\ &=N-g(n)=(N^{1/h}-1)^h. \end{aligned}$$

Hence

$$\begin{aligned} m^f+((n-m^f)^{1/h}+1)^h &= n+((n-m^f)^{1/h}+1)^h-(n-m^f) \\ &< n+N-(N^{1/h}-1)^h=n+g(n), \end{aligned}$$

since $((n-m^f)^{1/h}+1)^h-(n-m^f)$ is a strictly increasing function of $n-m^f$. This completes the proof.

References

[1] R. P. Bambah and S. Chowla: On numbers which can be expressed as a sum of two squares. Proc. Nat. Inst. Sci. India, **13**, 101-103 (1947).

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[3] S. Uchiyama: On the distribution of integers representable as a sum of two h-th powers. J. Fac. Sci., Hokkaidô Univ., Ser. I, **18**, 124-127 (1965).