136. *On a Theorem for M-Spaces*

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1. Introduction. Prof. K. Morita [4] has introduced the notion of M-spaces. He calls a topological space X an M-space if there exists a normal sequence $\{\mathfrak{U}_i \mid i=1, 2, \cdots\}$ of open coverings of X satisfying the condition $(*)$ below:

$(*)$ If a family $\mathfrak{K}$ consisting of a countable number of subsets of X has the finite intersection property and contains as a member a subset of $S(x_0, \mathfrak{U}_i)$ for every i and for some fixed point x_0 of X, then $\cap\{\bar{K} \mid K \in \mathfrak{K}\} \neq \emptyset$.

Recently, T. Kandô [2] has proved the following theorem.

Theorem 1. *Let $\{A_\alpha\}$ be a locally finite covering of a Hausdorff space X and let each A_α be a closed G_δ-subset of X. If each A_α is a normal M-space with respect to its relative topology, then the whole space X is also a normal M-space.*

In this connection he raised a problem whether Theorem 1 is valid without the G_δ-condition for A_α [2, p. 1053].

The purpose of this note is to give an affirmative answer to this problem; namely, we shall prove the following theorem.

Theorem 2. *Let $\{A_\alpha\}$ be a locally finite closed covering of a Hausdorff space X. If each A_α is a normal M-space with respect to its relative topology, then the whole space X is also a normal M-space.*

Most terminologies and notations used in this note are the same as those of J. W. Tukey [7].

We are indebted to Prof. K. Morita for valuable advices and encouragements throughout this study.

2. Lemmas. Lemma 1. *Let $\{A_i \mid i=1, 2\}$ be a binary closed covering of a Hausdorff space X. If each A_i is a normal M-space, then X is a normal M-space.*

Proof. According to a result of A. Okuyama [6] each A_i is collectionwise normal and countably paracompact, and hence by K. Morita [5] the whole space X is also collectionwise normal and countably paracompact.

Suppose that $\mathfrak{B}(A_1) \cap \mathfrak{B}(A_2) = \emptyset$.[1] Then we have $\mathfrak{J}(A_1) \cup \mathfrak{J}(A_2) = X$.[2] Since X is normal there exist two closed G_δ-subsets F_1 and

1) $\mathfrak{B}(A)$ means the boundary of a set A.

2) $\mathfrak{J}(A)$ means the interior of a set A.

F_2 such that $F_1\cup F_2=X$ and $F_i\subset\mathfrak{J}(A_i)\subset A_i$ $(i=1, 2)$. Hence by Theorem 1 X is a normal M-space. We may assume, therefore, that $\mathfrak{B}(A_1)\cap\mathfrak{B}(A_2)\neq\emptyset$.

Now, by assumption each A_i is an M-space, and hence there exists a normal sequeuce $\{\mathfrak{U}_n^{(i)}\mid n=1, 2, \cdots\}$ of open coverings of the subspace A_i satisfying the condition $(*)$ with respect to the subspace A_i. Moreover, we can assume that each covering $\mathfrak{U}_n^{(i)}$ is locally finite in A_i by [2, Lemma 1].

Let us put

$$\mathfrak{U}_n=\{U\cap\mathfrak{B}(A_1)\cap\mathfrak{B}(A_2)\mid U\in\mathfrak{U}_n^{(1)}\}\wedge\{U\in\mathfrak{B}(A_1)\cap\mathfrak{B}(A_1)\mid U\in\mathfrak{U}_n^{(2)}\}$$

for every positive integer n. Then $\mathfrak{U}_n$ is a locally finite open covering of $\mathfrak{B}(A_1)\cap\mathfrak{B}(A_2)$; for the sake of simplicity we shall denote the members of $\mathfrak{U}_n$ by $U_{n\alpha}$, $\alpha\in\Omega_n$, that is, $\mathfrak{U}_n=\{U_{n\alpha}\mid\alpha\in\Omega_n\}$.

(I). By a theorem of C. H. Dowker [1] there exists a locally finite open family $\mathfrak{V}_1=\{V_{1\alpha}\mid\alpha\in\Omega_1\}$ of X such that $U_{1\alpha}\supset(\mathfrak{B}(A_1)\cap\mathfrak{B}(A_2))\cap V_{1\alpha}$ for any $\alpha\in\Omega_1$. Moreover, we can assume $V_{1\alpha}\cap A_i\subset U^{(i)}$ for some $U^{(i)}\in\mathfrak{U}_1^{(i)}$ $(i=1, 2;\ \alpha\in\Omega_1)$. Since $\mathfrak{B}(A_1)\cap\mathfrak{B}(A_2)$ is contained in an open set $\mathfrak{V}_1^*$,[3] there exists an open set G_1 such that $\mathfrak{B}(A_1)\cap\mathfrak{B}(A_2)\subset G_1\subset\bar{G}_1\subset\mathfrak{V}_1^*$. Let us put $\mathfrak{V}_1^{(i)}=\{U\cap\mathfrak{J}(A_i)\cap(X-\bar{G}_1)\mid U\in\mathfrak{U}_1^{(i)}\}(i=1,2)$ then $\mathfrak{W}_1=\mathfrak{V}_1^{(1)}\cup\mathfrak{V}_1^{(2)}\cup\mathfrak{V}_1$ is a locally finite open covering of X.

(II). Since $\mathfrak{U}_2$ is a locally finite open covering of $\mathfrak{B}(A_1)\cap\mathfrak{B}(A_2)$, by C. H. Dowker [1] there exists a locally finite open fiamily $\mathfrak{V}_2=\{V_{2\alpha}\mid\alpha\in\Omega_2\}$ of X such that $U_{2\alpha}\supset\mathfrak{B}(A_1)\cap\mathfrak{B}(A_2)\cap V_{2\alpha}$ for every $\alpha\in\Omega_2$. Moreover we can assume that $V_{2\alpha}\cap A_i\subset U^{(i)}$ for some $U^{(i)}\in\mathfrak{U}_2^{(i)}(i=1, 2;\ \alpha\in\Omega_2)$ and $V_{2\alpha}\subset S(\mathfrak{B}(A_1)\cap\mathfrak{B}(A_2), \mathfrak{W}_1)\cap G_1$ for every $\alpha\in\Omega_2$. Since $\mathfrak{B}(A_1)\cap\mathfrak{B}(A_2)$ is contained in an open set $\mathfrak{V}_2^*$, there exists an open set G_2 such that $\mathfrak{B}(A_1)\cap\mathfrak{B}(A_2)\subset G_2\subset\bar{G}_2\subset\mathfrak{V}_2^*$. Let us put $\mathfrak{V}_2^{(i)}=\{U\cap\mathfrak{J}(A_i)\cap(X-\bar{G}_2)\mid U\in\mathfrak{U}_2^{(i)}\}(i=1, 2)$, then $\mathfrak{V}_2^{(1)}\cup\mathfrak{V}_2^{(2)}\cup\mathfrak{V}_2$ is a locally finite open covering of X. There exists a locally finite open covering $\mathfrak{W}_2$ such that $\mathfrak{W}_2$ is a star-refinement of $\mathfrak{W}_1$ and $\mathfrak{V}_2^{(1)}\cup\mathfrak{V}_2^{(2)}\cup\mathfrak{V}_2$.

(III). Now, by the same procedure as in case (II) we can easily construct by induction with respect to $n(n>2)$ a locally finite open family $\mathfrak{V}_n=\{V_{n\alpha}\mid\alpha\in\Omega_n\}$ of X, an open set G_n, a locally finite open family $\mathfrak{V}_n^{(i)}$ of $X(i=1, 2)$ and a locally finite open covering $\mathfrak{W}_n$ of X which satisfy the following conditions.

(1) $U_{n\alpha}\supset(\mathfrak{B}(A_1)\cap\mathfrak{B}(A_2))\cap V_{n\alpha}$, $(\alpha\in\Omega_n)$.

(2) $V_{n\alpha}\cap A_i\subset U^{(i)}$ for some $U^{(i)}\in\mathfrak{U}_n^{(i)}$, $(i=1, 2;\ \alpha\in\Omega_n)$.

(3) $V_{n\alpha}\subset S(\mathfrak{B}(A_1)\cap\mathfrak{B}(A_2), \mathfrak{W}_{n-1})\cap G_{n-1}$, $(\alpha\in\Omega_n)$.

(4) $\mathfrak{B}(A_1)\cap\mathfrak{B}(A_2)\subset G_n\subset\bar{G}_n\subset\mathfrak{V}_n^*$.

(5) $\mathfrak{V}_n^{(i)}=\{U\cap\mathfrak{J}(A_i)\cap(X-\bar{G}_n)\mid U\in\mathfrak{U}_n^{(i)}\}$, $(i=1, 2)$.

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3) $\mathfrak{V}^*$ means the sum of elements of a family $\mathfrak{V}$.

(6) $\mathfrak{V}_n^{(1)} \cup \mathfrak{V}_n^{(2)} \cup \mathfrak{V}_n$ is a locally finite open covering of X.

(7) $\mathfrak{W}_n$ is a star-refinement of $\mathfrak{W}_{n-1}$ and $\mathfrak{V}_n^{(1)} \cup \mathfrak{V}_n^{(2)} \cup \mathfrak{V}_n$.

(IV). We shall prove that the normal sequence $\{\mathfrak{W}_n\}$ satisfies the M-space condition $(*)$. To prove this, let $\mathfrak{K}=\{K_n \mid n=1, 2, \cdots\}$ be any family consisting of a countable number of subsets of X having the finite intersection property and suppose that $\mathfrak{K}$ contains as its member a subset K_{k_n} of $S(x_0, \mathfrak{W}_n)$ for every n and for some fixed point x_0 of X. We have to show $\cap\{\bar{K} \mid K \in \mathfrak{K}\} \neq \emptyset$. Without loss of generality we may assume that for every positive integer n $K_n \supset K_{n+1}$. We distinguish the following three cases.

(i) $x_0 \notin \bigcap_{n=1}^{\infty} S(\mathfrak{V}(A_1) \cap \mathfrak{V}(A_2), \mathfrak{W}_n)$.

(ii) $x_0 \in \bigcap_{n=1}^{\infty} S(\mathfrak{V}(A_1) \cap \mathfrak{V}(A_2), \mathfrak{W}_n)$, $x_0 \in A_i$ and $\{K_n \cap A_i \mid K_n \in \mathfrak{K}\}$ has the finite intersection property for some i.

(iii) $x_0 \in \bigcap_{n=1}^{\infty} S(\mathfrak{V}(A_1) \cap \mathfrak{V}(A_2), \mathfrak{W}_n)$, $x_0 \in A_i$ and $\{K_n \cap A_j \mid K_n \in \mathfrak{K}\}$ has the finite intersection property for some i and j with $i \neq j$.

Case (i). There exists a positive integer n_0 such that $x_0 \notin S(\mathfrak{V}(A_1) \cap \mathfrak{V}(A_2), \mathfrak{W}_{n_0})$. By (3) $\mathfrak{V}_{n_0+1}^*$ is contained in $S(\mathfrak{V}(A_1) \cap \mathfrak{V}(A_2), \mathfrak{W}_{n_0})$. Hence x_0 is not contained in $\mathfrak{V}_{n_0+1}^*$. On the other hand, for every $n > n_0+1$ there exists an element W of $\mathfrak{W}_{n-1}$ such that $K_{k_n} \subset S(x_0, \mathfrak{W}_n) \subset W$. This set W is contained in an element $U_{n-1}^{(i)}$ of $\mathfrak{V}_{n-1}^{(i)}$ for some $i (i=1$ or $2)$ since $x_0 \notin \mathfrak{V}_{n-1}^*$ (i varies with n). Namely, we have $K_{k_n} \subset U_{n-1}^{(i)}$ for $i=1$ or 2. Without loss of generality we may assume that there are infinitely many $n > n_0+1$ such that $K_{k_n} \subset U_{n-1}^{(1)}$. Hence we have $K_{k_n} \subset S(x_0, \mathfrak{U}_n^{(1)})$ for infinitely many n with $n > n_0+1$. From the assumption that A_1 is a closed M-space in X it follows that $\bigcap_{n=1}^{\infty} \bar{K}_n = \emptyset$.

Case (ii). Without loss of generality we assume that $i=1$. By conditions (7) and (8) we have

$$S(x_0, \mathfrak{W}_{n+2}) \subset S(\mathfrak{V}(A_1) \cap \mathfrak{V}(A_2), \mathfrak{W}_{n+1}) \subset S(\mathfrak{V}(A_1) \cap \mathfrak{V}(A_2), \mathfrak{V}_{n+1}) \subset G_n.$$

Hence by (5) we obtain $S(x_0, \mathfrak{W}_{n+2}) \cap (\mathfrak{V}_n^{(1)*} \cup \mathfrak{V}_n^{(2)*}) = \emptyset$ and so, x_0 is not contained in $\mathfrak{V}_n^{(1)*} \cup \mathfrak{V}_n^{(2)*}$. Thus, we have $K_{k_{n+2}} \subset S(x_0, \mathfrak{W}_{n+2}) \subset S(x_0, \mathfrak{V}_n^{(1)} \cup \mathfrak{V}_n^{(2)} \cup \mathfrak{V}_n) = S(x_0, \mathfrak{V}_n)$. Consequently from (2) it follows that $K_{k_{n+2}} \cap A_1 \subset S(x_0, \mathfrak{U}_n^{(1)})$. Since A_1 is a closed M-space and $\{K_n \cap A_1\}$ has the finite intersection property, we have $\bigcap_{n=1}^{\infty} \{\overline{K_n \cap A_1}\} \neq \emptyset$. Therefore, of course, $\bigcap_{n=1}^{\infty} \bar{K}_n \neq \emptyset$.

Case (iii). Without loss of generality we assume that $i=1, j=2$. Let us put $S(x_0, \mathfrak{W}_n) \cap \mathfrak{V}(A_1) \cap \mathfrak{V}(A_2) = M_n$ for every positive integer n; then $\{M_n \mid n=1, 2, \cdots\}$ has the finite intersection property and satisfies $M_n \supset \bar{M}_{n+1}$ for every n. Since $\{M_n\}$ satisfies the assumption of $(*)$ in the M-space A_1, we have $\bigcap_{n=1}^{\infty} \bar{M}_n = \bigcap_{n=1}^{\infty} M_n \neq \emptyset$. Let x' be a point of $\bigcap_{n=1}^{\infty} M_n$. Then we shall show that $S(x', \mathfrak{U}_n^{(2)}) \supset A_2 \cap K_{k_{n+1}}$. There exists an element W of $\mathfrak{W}_n$ such that $S(x_0, \mathfrak{W}_{n+1}) \subset W$. By the assumption we have $W \cap \mathfrak{V}(A_1) \cap \mathfrak{V}(A_2) \neq \emptyset$. Hence there exists $V_{n\alpha} \in \mathfrak{V}_n$ such that $W \subset V_{n\alpha}$. From (2) it follows that $V_{n\alpha} \cap A_2 \subset U_n^{(2)}$ for some $U_n^{(2)} \in \mathfrak{U}_n^{(2)}$. Therefore, we have $U_n^{(2)} \supset V_{n\alpha} \cap A_2 \supset S(x_0, \mathfrak{W}_{n+1}) \cap A_2 \ni x'$. Consequently, $S(x', \mathfrak{U}_n^{(2)}) \supset K_{k_{n+1}} \cap A_2$. Since A_2 is a closed M-space in X we have $\bigcap_{n=1}^{\infty} \overline{K_n \cap A_2} \neq \emptyset$ and hence $\bigcap_{n=1}^{\infty} \bar{K}_n = \emptyset$. Thus Lemma 1 is completely proved.

From Lemma 1 we have the following lemma as an immediate consequence.

Lemma 2. *Let $\{A_1, A_2, \cdots, A_n\}$ be a finite closed covering of a Hausdorff space X. If each A_i is a normal M-space, then X is a normal M-space.*

Lemma 3. *Let $\{A_\alpha\}$ be a locally finite closed covering of a Hausdorff space X and suppose that the order of $\{A_\alpha\}$ does not exceed n. If each A_α is a normal M-space then X is a normal M-space.*

Proof. From the assumption it is seen that X is collectionwise normal and countably paracompact (cf. K. Morita [5]). By a theorem of M. Katětov [3] and the normality of X there exists a locally finite closed covering $\{F_\alpha\}$ of X such that each F_α is a G_δ-subset of X, $A_\alpha \subset F_\alpha$ for each α and $\{F_\alpha\}$ is similar to $\{A_\alpha\}$. Let $\{\alpha_1, \cdots, \alpha_n\}$ be a set of n distinct indices. Then we have $F_{\alpha_1} \cap \cdots \cap F_{\alpha_n} \subset A_{\alpha_1} \cup \cdots \cup A_{\alpha_n}$, since the order of $\{A_\alpha\}$ does not exceed n. By Lemma 2 $A_{\alpha_1} \cup \cdots \cup A_{\alpha_n}$ is a normal M-space, therefore, $F_{\alpha_1} \cap \cdots \cap F_{\alpha_n}$ is a normal M-space which is a closed G_δ-subset of X. For any set $\{\alpha_1, \cdots, \alpha_r\}$ of r distinct indices we denote

$$\Omega(\alpha_1, \cdots, \alpha_r) = \{\alpha \mid F_\alpha \cap (F_{\alpha_1} \cap \cdots \cap F_{\alpha_r}) \neq \emptyset, \alpha \neq \alpha_i (i=1, 2, \cdots, r)\}.$$

We can easily prove the following relation:

$$(**) \quad F_{\alpha_1} \cap \cdots \cap F_{\alpha_{n-1}} \subset \cup \{F_\alpha \cap F_{\alpha_1} \cap \cdots \cap F_{\alpha_{n-1}} \mid \alpha \in \Omega(\alpha_1, \cdots \alpha_{n-1})\} \cup (A_{\alpha_1} \cup \cdots \cup A_{\alpha_{n-1}}).$$

Since $\{F_\alpha \cap F_{\alpha_1} \cap \cdots \cap F_{\alpha_{n-1}} \mid \alpha \in \Omega(\alpha_1, \cdots, \alpha_{n-1})\}$ is a locally finite family of closed G_δ-subsets of X, by Theorem 1 $\cup \{F_\alpha \cap F_{\alpha_1} \cap \cdots \cap F_{\alpha_{n-1}} \mid \alpha \in \Omega(\alpha_1, \cdots, \alpha_{n-1})\}$ is a normal M-space. Hence the right side of $(**)$ is a normal M-space. Therefore $F_{\alpha_1} \cap \cdots \cap F_{\alpha_{n-1}}$ is a normal M-space and closed G_δ-subset of X. We can prove successively that for r distinct indices $\alpha_1, \cdots, \alpha_r F_{\alpha_1} \cap \cdots \cap F_{\alpha_r}$ is a closed G_δ-subset and a normal M-space, and $F_{\alpha_1} \cap \cdots \cap F_{\alpha_{r-1}} \subset \cup \{F_\alpha \cap F_{\alpha_1} \cap \cdots \cap F_{\alpha_{r-1}} \mid \alpha \in \Omega(\alpha_1, \cdots, \alpha_{r-1})\} \cup (A_{\alpha_1} \cup \cdots \cup A_{\alpha_{r-1}})$. Finally, we obtain that each F_α is a closed G_δ-subset of X and a normal M-space. From Theorem 1 it follows now that X is a normal M-space. The proof of Lemma 3 is thus completed.

3. Proof of Theorem 2. Let us put

$$G_n = \{x \mid x \text{ is contained in at most } n \text{ members of } \{A_\alpha\}\}$$
$$\text{for every positive integer } n.$$

Then $\{G_n\}$ is a countable open covering of X. Since X is normal and countably paracompact, there exists a locally finite closed covering $\{B_n\}$ of X such that each B_n is a G_δ-subset of X and $\{B_n\}$ refines $\{G_n\}$. As $\{B_n \cap A_\alpha \mid \alpha\}$ is a locally finite closed conering of B_n such that the order of $\{B_n \cap A_\alpha \mid \alpha\}$ is finite, by Lemma 3 B_n is a closed G_δ-subset of X and a normal M-space. Therefore, from Theorem 1 X is a normal M-space. Thus we see that Theorem 2 holds.

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