135. *A Remark on a Class of Operators*

By Takayuki FURUTA,*) Midori HORIE,**)
and Ritsuo NAKAMOTO***)

(Comm. by Kinjirô KUNUGI, M.J.A., Sept. 12, 1967)

1. Following after Istratescu, a (bounded linear) operator T acting on a Hilbert space $\mathfrak{H}$ is of *class* (N) or *paranormal* in the sense of [2], symbolically $T \in \mathfrak{P}$, if

(1) $$\| T^2 x \| \geqq \| Tx \|^2,$$

for any $x \in H$ with $\| x \| = 1$. Incidentally, it is noteworthy that the definition of paranormality is applicable for operators on general Banach spaces.

A hyponormal operator in the sense of Berberian [1] is paranormal and a paranormal operator T is a *normaloid* in the sense that T satisfies

(2) $$\| T^n \| = \| T \|^n, \qquad n = 1, 2, 3, \cdots,$$

which are pointed out by Istratescu, Saito, and Yoshino [3], cf. also Stampfli [5].

We shall prove the following

Theorem. *If a paranormal operator T has a compact power T^k, then T is compact. However, this is not true for normaloid operators in general.*

The first half of the theorem for hermitean operators is already pointed out by Schatten [4; p. 18].

2. Conveniently, we shall here introduce a new notion: An operator T is *k-paranormal*, symbolically $T \in \mathfrak{P}_k$ $(k > 0)$, if T satisfies

(3) $$\| T^{k+1} x \| \geqq \| Tx \|^{k+1},$$

for any $x \in \mathfrak{H}$ with $\| x \| = 1$. Obviously, $\mathfrak{P}$ coincides with $\mathfrak{P}_1$. Moreover, we have

(4) $$T \in \mathfrak{P} \rightarrow T \in \mathfrak{P}_k, \qquad k > 0.$$

(4) is already established in [3]. However, for the sake of completeness, we shall reproduce the proof of (4). For $k = 1$, (4) is trivial. If (4) is true for $k - 1$, then we have

$$\| T^{k+1} x \| = \| Tx \| \left\| T^k \frac{Tx}{\| Tx \|} \right\| \geqq \| Tx \| \left\| T \frac{Tx}{\| Tx \|} \right\|^k$$
$$= \frac{\| T^2 x \|^k}{\| Tx \|^{k-1}} \geqq \frac{\| Tx \|^{2k}}{\| Tx \|^{k-1}} = \| Tx \|^{k+1},$$

*) Ibaraki University, Hitachi.
**) Osaka Kyoiku University.
***) Tennoji Senior High School.

which proves (4).

By (4), we shall prove the following (5) instead of the first half of the theorem:

$$(5) \qquad T \in \mathfrak{P}_{k-1}, \quad T^k \in \mathfrak{C} \rightarrow T \in \mathfrak{C},$$

where $\mathfrak{C}$ is the algebra of all compact operators.

Let us suppose that

$$x_\alpha \rightarrow 0 (\text{weakly}), \qquad \| x_\alpha \| \leqq 1.$$

Since $T \in \mathfrak{P}_{k-1}$, (3) implies

$$\| T^k x_\alpha \| \geqq \frac{\| Tx_\alpha \|^k}{\| x_\alpha \|^{k-1}} \geqq \| Tx_\alpha \|^k,$$

which tells us that Tx_α converges strongly to 0, since $\| T^k x_\alpha \| \rightarrow 0$ by the compactness of T^k. Therefore, T is compact.

3. To prove the remainder half of the theorem, let us put $\mathfrak{H} = (l^2)$. Define an operator T by

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

with respect to the orthonormal basis

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \cdots$$

Then, wen can easily deduce

$$(6) \qquad Te_i = \begin{cases} e_1 & (i=1) \\ e_{i+1} & (i=2j) \qquad j=1, 2, \cdots \\ 0 & (i=2j+1), \end{cases}$$

Hence

$$\| T \| = 1 \quad \text{and} \quad T^k = P \quad (k \geqq 2),$$

where P is the projection belonging to the subspace spanned by the scalar multiples of e_1. Therefore,

$$\| T^k \| = 1 = \| T \|^k$$

for all k, which shows that T *is a normaloid.*

Since $T^k = P$ for $k \geqq 2$, T^k *is compact for* $k \geqq 2$, whereas T *is not compact* since the range of T contains an infinite orthonormal set $\{e_i;\ i = 1, 3, 5, 7, \cdots\}$. The second half of the theorem is now proved.

4. At this end, we shall list a few remarks.

(a) The above example also tells us that there exists a non-normal normaloid T with the compact squre $T^2=P$. Cf. also [2].

(b) It may be noticed that the first half of the theorem has a proof based on [3, Theorem 2] since the hypothesis of the theorem implies that T is normal. However, our proof is simpler and applicable for Banach spaces.

References

[1] S. K. Berberian: Introduction to Hilbert Space. Oxford Univ. Press, New York (1961).

[2] T. Furuta: On the class of paranormal operators. Proc. Japan Acad., **43**, 594—598 (1967).

[3] V. Istratescu, T. Saito, and T. Yoshino: On a class of operators. Tohoku Math. J., **18**, 410-412 (1966).

[4] R. Schatten: Norm Ideals of Completely Continuous Operators. Springer, Berlin (1960).

[5] J. G. Stampfli: Hyponormal operators. Pacif. J. Math., **12**, 1453-1458 (1962).