133. *A Characterization of Spectraloid Operators and its Generalization*

By Takayuki FURUTA and Zirô TAKEDA
Faculty of Engineering, Ibaraki University

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Normaloid operators are characterized by the equality $\|T^n\| = \|T\|^n$ for every natural number n. We give here a similar characterization of spectraloid operators and coincidently we define two families of new classes of non-normal operators broader than the class of normaloid operators associating with these characterizations. Each family forms an atomic lattice by the set inclusion relation.

In what follows an operator means a bounded linear operator on a complex Hilbert space.

1. For each operator T we associate three non-negative numbers

$$\|T\| = \sup_{\|x\|=\|y\|=1} |(Tx, y)|, \qquad \|T\|_N = \sup_{\|x\|=1} |(Tx, x)|,$$

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\},$$

(where $\sigma(T)$ is the spectrum of T), which are called the operator norm, numerical radius and the spectral radius of T respectively. These are related by

$$(1) \qquad r(T) \leqq \|T\|_N \leqq \|T\|$$

$$(2) \qquad r(T) = \lim_{n\to\infty} \|T^n\|_N^{\frac{1}{n}} = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$$

For $\|T\|_N$ the following properties are known

$$(3) \qquad \|T\|_N = 0 \quad \text{if and only if } T = 0,$$

$$(4) \qquad \|\lambda T\|_N = |\lambda|\, \|T\|_N \quad \text{for every scalar } \lambda,$$

$$(5) \qquad \|T+S\|_N \leqq \|T\|_N + \|S\|_N$$

$$(6) \qquad 1/2\|T\| \leqq \|T\|_N \leqq \|T\| \,.$$

That is, $\|T\|_N$ is a new norm equivalent to the operator norm $\|T\|$. On the other hand $r(T)$ satisfies (4) but not (3) and (5) remains only in a restricted form. Hence $r(T)$ is not a norm in a strict sense but we may interpret it as a kind of generalized norm.

It is known that these satisfy the same kind of power inequality:

$$(7) \qquad \|T^n\| \leqq \|T\|^n, \qquad \|T^n\|_N \leqq \|T\|_N^n, \qquad r(T^n) \leqq r(T)^n,$$

Exactly $r(T^n) = r(T)^n$ for every operator T by the spectral mapping theorem ([1]—[4]).

Following Halmos [2] and Wintner [5], we give

Definition 1. An operator T is called to be spectraloid if

$$\|T\|_N = r(T)$$

Definition 2. An operator T is called to be normaloid if

$$\|T\|=r(T)$$

Clearly by (1) every normaloid operator is spectraloid but the inverse implication is not true.

It is known that T is normaloid if and only if $\|T\|=\|T\|_N$, and this is equivalent to the condition

(8) $\|T^n\|=\|T\|^n$ for all positive integer n.

We give here an analogous characterization for spectraloid operators.

Theorem 1. *T is a spectraloid operator if and only if*

(9) $$\|T^n\|_N=\|T\|_N^n.$$

Thus corresponding to $r(T)$, $\|T\|_N$, $\|T\|$ satisfying $r(T)\leqq\|T\|_N\leqq\|T\|$ we get a parallelism about power equalities as follows.

(A) $r(T^n)=r(T)^n$ *for every operator T*

(B) $\|T^n\|_N=\|T\|_N^n$ *if and only if T is spectraloid*

(C) $\|T^n\|=\|T\|^n$ *if and only if T is normaloid.*

Proof of Theorem 1. If T is spectraloid, then

$$\|T\|_N^n=r(T)^n=r(T^n)\leqq\|T^n\|_N,$$

the reverse inequality is due to the power inequality, so $\|T^n\|_N=\|T\|_N^n$.

If $\|T^n\|_N=\|T\|_N^n$ holds, then

$$\|T\|_N^n=\|T^n\|_N\leqq\|T^n\|, \qquad \|T\|_N\leqq\|T^n\|^{\frac{1}{n}},$$

$$\text{so } \|T\|_N\leqq\lim_{n\to\infty}\|T^n\|^{\frac{1}{n}}=r(T),$$

thus $\|T\|_N=r(T)$ because the reverse inequality is valid by (1).

Corollary 1. *If T is a spectraloid operator, then T^n is also spectraloid for every positive integer n.*

Proof. Let T be spectraloid, then the following equality holds by the above Theorem,

$$\|T^n\|_N=\|T\|_N^n=r(T)^n=r(T^n).$$

thus T^n is also spectraloid.

Corollary 2. *Spectraloid quasinilpotent operator is identically* 0.

Proof. If T is spectraloid quasinilpotent operator, the following equality holds,

$$\|T\|_N=r(T)=\lim_{n\to\infty}\|T^n\|^{\frac{1}{n}}=0.$$

So T is identically 0 by (3).

We notice here the validity of the proof of Theorem 1 essentially depends upon the intermediate property of the numerical radius $r(T)\leqq\|T\|_N\leqq\|T\|$ and the power inequality $\|T^n\|_N\leqq\|T\|_N^n$. This fact gives us the basis of the argument in subsequent sections.

2. For an operator T, put

$$\|T\|_p=\sup_{\|x\|=\|y\|=1}|(T^px, y)|^{\frac{1}{p}}=\|T^p\|^{\frac{1}{p}},$$

where p is a natural number. Then clearly

(10) $\|T\|_p=0$ if and only if $T^p=0$.

(11) $\|\lambda T\|_p=|\lambda|\cdot\|T\|_p$, where λ is a scalar,

(12) $\|T^n\|_p\leqq\|T\|_p^n$ (power inequality)

because $\|T^n\|_p=\|T^{np}\|^{\frac{1}{p}}\leqq\|T^p\|^{\frac{n}{p}}\leqq\|T\|_p^n$.

(13) $r(T)\leqq\|T\|_p\leqq\|T\|$ (intermediate property)

because $r(T)^p=r(T^p)\leqq\|T^p\|\leqq\|T\|^p$.

(14) $\lim_{p\to\infty}\|T\|_p=r(T)$.

Thus $\|T\|_p$ is not a norm but it has many similar properties to the spectral radius or the operator norm.

Definition 3. An operator T is called to be *power p normaloid* or simply *N_p-operator* if

(15) $\|T\|_p=r(T)$.

We put the class of the power p normaloid operators by N_p and the family of N_p by $\mathfrak{N}$: $\mathfrak{N}=\{N_p\}$. Clearly the class N_1 is the set of normaloid operators and it is the smallest class in the family $\mathfrak{N}$. By the intermediate property (13) and the power inequality (12), the proof of theorem 1 is valid for these classes N_p and we get

Theorem 2. *T is a N_p-operator if and only if*

(16) $\|T^n\|_p=\|T\|_p^n$.

Corollary 1′. *If T is an N_p-operator, then T^n is also an N_p-operator for every positive n.*

Corollary 2′. *If T is a quasinilpotent N_p-operator, then $T^p=0$.*

By (14) $\lim_{p\to\infty}\|T\|_p=r(T)$ and so we may naturally put $r(T)=\|T\|_\infty$. Since $r(T^n)=r(T)^n$ for every operator T, we may interpret N_∞ as the whole set of operators. Theorem 2 shows the parallelism stated in §1 remains completely for intermediate classes N_p between N_1 and N_∞.

The nilpotent operator $T=\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$ given by the $n\times n$ matrix

(17) $$M=\begin{pmatrix} 0 & 0 & 0\cdots 0 & 0 \\ 1 & 0 & 0\cdots 0 & 0 \\ 0 & 1 & 0\cdots 0 & 0 \\ 0 & 0 & 1\cdots 0 & 0 \\ & & \ddots & \\ 0 & 0 & 0\cdots 1 & 0 \end{pmatrix}$$

is clearly an N_p-operator for $p\geqq n$ but not for $p<n$. Hence N_p *are mutually distinctive classes for different indices.*

Theorem 3. *$N_p\subset N_q$ if and only if p divides q.*

Proof. If p divides q, for $T\in N_p$

$$r(T)\leqq\|T\|_q=\|T^q\|^{\frac{1}{q}}=\|T^{pl}\|^{\frac{1}{pl}}\leqq\|T^p\|^{\frac{1}{p}}=r(T), \qquad (q=pl).$$

Therefore $T\in N_q$.

Now let M' be the $n\times n$ matrix

$$(18)\qquad M'=\begin{pmatrix} 0 & 0 & 0\cdots 0 & 2 \\ 1 & 0 & 0\cdots 0 & 0 \\ 0 & 1 & 0\cdots 0 & 0 \\ 0 & 0 & 1\cdots 0 & 0 \\ \vdots & \vdots & \vdots \quad \vdots & \vdots \\ 0 & 0 & \cdots\cdots 1 & 0 \end{pmatrix}$$

Clearly

$$\|M'\|=\|M'^2\|=\cdots=\|M'^n\|=2,$$
$$\|M'^{n+1}\|=\|M'^{n+2}\|=\cdots=\|M'^{2n}\|=4.$$

In general $\|M'^l\|=2^s$ for l such that $(s-1)n<l\leqq sn$, and $r(T)=2^{\frac{1}{n}}$. Hence

$$(19)\qquad T=\begin{pmatrix} M' & 0 & 0 & \cdots \\ 0 & M' & 0 & \cdots \\ 0 & 0 & M' & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix}$$

is in N_p only for $p=sn$ $(s=1, 2, \cdots)$ and $T\notin N_p$ for other p. Thus $N_p\not\subseteq N_q$ if p does not divide q. q.e.d

Especially $M'=\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ gives T which belongs only to N_p's with even indices.

Corollary 3. *The family of classes $\mathfrak{N}=\{N_p \mid p=1, 2, \cdots \infty\}$ forms an atomic lattice by the inclusion relation. The greatest element and the least element are N_∞ and N_1 respectively and atomic elements are N_p's with prime indices.*

An operator T is called to be convexoid if the closure of numerical range $\overline{W(T)=\{(Tx, x): \|x\|=1\}}$ equals to the convex hull of the spectrum $\sigma(T)$ of T.

The class of convexoid operators is not contained in the class of normaloids and vice versa, but they are both contained in the class of spectraloids.

Now let

$$T=\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix},$$

where M is the $n\times n$ matrix given by (17) and N is a normaloid and convexoid operator such that the spectrum $\sigma(N)$ is the disc with radius ρ (ρ is the numerical radius of M, $\rho<1$). For example we may take $N=\rho U$, where U is the unilateral shift operator. Then every power of T is always convexoid but T^l is normaloid only if $l\geqq n$. Hence we know that for any natural number p there exists a convexoid operator which does not contained in N_p. In other words, *the class of convexoids is not contained in any* N_p *with finite index* p *and similar for the class of spectraloids.*

The union $U_{p=1,2,\dots}N_p$ *is not equal to* N_∞. For example T given by (19) taking $M'=\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}$ is not contained in this union.

3. Taking $\|T\|_N$ instead of $\|T\|$, we can proceed parallel to § 2. We define

$$\|T\|_{N,p}=\sup_{\|x\|=1}|(T^p x, x)|^{\frac{1}{p}}=\|T^p\|_N^{\frac{1}{p}}.$$

Then

(20) $$\|T\|_{N,p}=0 \qquad \text{if and only if } T^p=0.$$

(21) $$\|\lambda T\|_{N,p}=|\lambda|\,\|T\|_{N,p}, \qquad \text{where } \lambda \text{ is a scalar.}$$

(22) $$\|T^n\|_{N,p}\leqq\|T\|_{N,p}^n \qquad \text{(power inequality)}$$

(23) $$r(T)\leqq\|T\|_{N,p}\leqq\|T\|_N\leqq\|T\| \qquad \text{(intermediate property).}$$

By (6)

$$1/2\|T^p\|\leqq\|T^p\|_N\leqq\|T^p\|$$

Hence

(24) $$2^{-\frac{1}{p}}\|T\|_p\leqq\|T\|_{N,p}\leqq\|T\|_p.$$

Since $\lim_{p\to\infty}\|T\|_p=r(T)$, $\lim_{p\to\infty}2^{-\frac{1}{p}}=1$,

(25) $$\lim_{p\to\infty}\|T\|_{N,p}=r(T)=\|T\|_\infty.$$

As well known $\|T\|=\|T\|_N$ if and only if $\|T\|=r(T)$, hence

(26) $\|T\|_p=\|T\|_{N,p}$ if and only if $\|T\|_p=r(T)$, that is, $T\in N_p$

Definition 4. An operator T is called to be *power* p *spectraloid* or *simply* S_p*-operator* if

$$\|T\|_{N,p}=r(T).$$

We put S_p the class of power p spectraloids and by S_∞ the whole set of operators.

Theorem 2′. *T is a S_p-operator if and only if*

(27) $$\|T^n\|_{N,p}=\|T\|_{N,p}^n.$$

Corollary 1″. *If* $T\in S_p$, *then* $T^n\in S_p$.

Corollary 2″. *If* T *is a quasinilpotent* S_p*-operator, then* $T^p=0$.

From the inequality $\|T\|_{N,p}\leqq\|T\|_p$, we get $N_p\subset S_p$ and so S_p are mutually distinctive classes for different indeces.

Theorem 3′. $S_p\subset S_q$ *if and only if* p *divides* q.

Proof. Clearly if p divides q, then $S_p\subset S_q$.

Let T be the operator given by (19) taking the $p\times p$ matrix M' (18). The spectral radius $r(T)=2^{\frac{1}{p}}$. For the unit vector

$$x=\left(\frac{1}{\sqrt{p}}\frac{1}{\sqrt{p}}\frac{1}{\sqrt{p}},\cdots\frac{1}{\sqrt{p}},0,0,0,\cdots\right)$$

and $n=sp+l(l=1,2,\cdots,p)$

$$(T^n x, x) = 2^s\left(1 + \frac{l}{p}\right).$$

Since

$$2^x < 1 + x \qquad (0 < x < 1),$$

$$r(T^n) = 2^s 2^{\frac{l}{p}} < 2^s\left(1 + \frac{l}{p}\right) = (T^n x, x) \qquad \text{if } l \neq p.$$

That is, $r(T^n) = \|T^n\|_N$, if and only if p divides n. Hence if p does not divide q, $T \in S_p$ but $T \notin S_q$.

Corollary 3″. *The family of classes* $\mathcal{S} = \{S_p \mid p = 1, 2, \cdots, \infty\}$ *forms an atomic lattice by the set inclusion relation. The greatest element is the whole set of operators* S_∞, *the least element is the class of spectraloids* S_1 *and atomic elements are* S_p'*s with prime indices.*

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