130 On Ranked Spaces and Linearity

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Let E be a linear space over the real or complex numbers, where defined families of subsets $\mathfrak{B}_n(n=0, 1, 2, \cdots,)$ which satisfy following conditions:

(A) For every V in $\mathfrak{B}$, $0 \in V$ (where $\mathfrak{B}=\bigcup_{n=0}^{\infty}\mathfrak{B}_n$).

(B) For U, V in $\mathfrak{B}$ there is a W in $\mathfrak{B}$ such that $W \subseteq U \cap V$.

(a) For any U in $\mathfrak{B}$ and for any integer n, there is an m such that $m \geq n$, and a V in $\mathfrak{B}_m$ such that $V \subseteq U$.

(b) $E \in \mathfrak{B}_0$.

For each point x in E, we shall call $x+V$ a neighbourhood of x with rank n, when $V \in \mathfrak{B}_n$. Then E is a ranked space [1] with indicator ω_0. Furthermore, for any sequence $\{x_n\}$ in E, we have $\{\lim x_n\} \ni x$ [1] if and only if $\{\lim (x_n - x)\} \ni 0$. In fact, if $\{\lim x_n\} \ni x$, there exists a sequence of neighbourhoods of x, $\{v_n(x)\}$, such that

$$v_n(x)=x+V_n,\ V_n \in \mathfrak{B}_{\alpha_n},\ \alpha_n \uparrow \infty,\ v_n(x) \supseteq v_{n+1}(x),\ x_n \in v_n(x).$$

This implies that $V_n \supseteq V_{n+1}$, and therefore $\{\lim (x_n - x)\} \ni 0$. The converse is also obvious.

Now, we set following three axioms concerning the relation between the linear operations and the ranks of neighbourhoods.

(1) There exists a non-negative function $\phi(\lambda, \mu)$, defined for $\lambda \geq 0$ and $\mu \geq 0$, such that $\lim_{\lambda, \mu \to \infty} \phi(\lambda, \mu)=\infty$, and the following holds; if $U \in \mathfrak{B}_l$, $V \in \mathfrak{B}_m$, $W \in \mathfrak{B}_n$, $n \leq \phi(l, m)$, and $U+V \subseteq W$, then, there is an $n^* \geq \phi(l, m)$, and a $W^* \in \mathfrak{B}_{n^*}$ such that $U+V \subseteq W^* \subseteq W$.

(2) There exists a non-negative function $\psi(\lambda, \mu)$, defined for $\lambda \geq 0$ and $\mu \geq 1$ such that $\lim_{\lambda \to \infty} \psi(\lambda, \mu)=\infty$ for each fixed μ, and the following holds; let α be a scalar with $|\alpha| \geq 1$. If $U \in \mathfrak{B}_m$, $V \in \mathfrak{B}_n$, $\alpha U \subseteq V$, and $n \leq \psi(m, |\alpha|)$, then there is an $n^* \geq \psi(m, |\alpha|)$ and a $V^* \in \mathfrak{B}_{n^*}$ such that $\alpha U \subseteq V^* \subseteq V$.

(3) Let $U \in \mathfrak{B}$ and $x \in U$. Then for any n, there is an $m \geq n$, a $V \in \mathfrak{B}_m$ and some positive ρ such that $\rho x \in V \subseteq U$.

Moreover, we assume that every V in $\mathfrak{B}$ is circled (i.e. if $x \in V$ and $|\alpha| \leq 1$, then $\alpha x \in V$).

When E satisfies all these axioms, we can assert that

I. if $\{\lim x_n\} \ni x$ and $\{\lim y_n\} \ni y$, then $\{\lim (x_n+y_n)\} \ni x+y$.

II. if $\{\lim x_n\} \ni x$, then for any scalar λ, $\{\lim \lambda x_n\} \ni \lambda x$.

III. if $\lim \lambda_n = \lambda$ (where λ_n, λ are scalars), then for any x in E, $\{\lim \lambda_n x\} \ni \lambda x$.

I. means the continuity of addition. II. and III. mean the continuity (more precisely, the separate continuity) of scalar multiplication.

Proof. Since $\{\lim x_n\} \ni x$ if and only if $\{\lim (x_n - x)\} \ni 0$, it suffices to show that, respectively,

I$'$. if $\{\lim x_n\} \ni 0$, and $\{\lim y_n\} \ni 0$, then $\{\lim (x_n + y_n)\} \ni 0$.

II$'$. if $\{\lim x_n\} \ni 0$, then for any λ, $\{\lim \lambda x_n\} \ni 0$.

III$'$. if $\lim \lambda_n = 0$, then for any x, $\{\lim \lambda_n x\} \ni 0$.

Proof of I$'$. From the hypothesis, there exist two sequences of neighbourhoods of 0, $\{U_n\}$, $\{V_n\}$, such that

$$U_n \in \mathfrak{V}_{\alpha_n},\ U_n \supseteq U_{n+1},\ \alpha_n \uparrow \infty,\ x_n \in U_n (n=1, 2, \cdots,)$$
$$V_n \in \mathfrak{V}_{\beta_n},\ V_n \supseteq V_{n+1},\ \beta_n \uparrow \infty,\ y_n \in V_n (n=1, 2, \cdots,)$$

Taking U_1, V_1, E, respectively, as U, V, W, and applying (1), we get an integer $\gamma_1^* \geq \phi(\alpha_1, \beta_1)$ and a $W_1^* \in \mathfrak{V}_{\gamma_1^*}$ with $U_1 + V_1 \subseteq W_1^*$. Then, clearly, $x_n + y_n \in W_1^*$ for any n. Since $\lim_{n\to\infty} \phi(\alpha_n, \beta_n) = \infty$, we can choose an $n_1 > 1$, such that $\phi(\alpha_{n_1}, \beta_{n_1}) > \gamma_1^*$. As $U_{n_1} + V_{n_1} \subseteq U_1 + V_1 \subseteq W_1^*$, we can apply again axiom (1) to U_{n_1}, V_{n_1}, W_1^*, and find a $\gamma_2^* \geq \phi(\alpha_{n_1}, \beta_{n_1})$ and a $W_2^* \in \mathfrak{V}_{\gamma_2^*}$ with $U_{n_1} + V_{n_1} \subseteq W_2^* \subseteq W_1^*$. It is clear that $\gamma_2^* > \gamma_1^*$ and $x_n + y_n \in W_2^*$ for $n \geq n_1$.

Continuing this process, we obtain sequences of integers, $\{n_i\}$, $\{\gamma_i^*\}$, and a sequence of sets $\{W_i^*\}$ such that $n_i < n_{i+1}$, $\gamma_i^* < \gamma_{i+1}^*$; $W_i^* \in \mathfrak{V}_{\gamma_i^*}$, $W_i^* \supseteq W_{i+1}^*$, and $x_n + y_n \in W_i^*$ when $n_{i-1} \leq n (i=1, 2, \cdots,)$, where $n_0 = 1$. Now, put $\gamma_n = \gamma_i^*$, $W_n = W_i^*$ when $n_{i-1} \leq n < n_i (i=1,2,\cdots,)$. Then, $W_n \in \mathfrak{V}_{\gamma_n}$, $W_n \supseteq W_{n+1}$, $\gamma_n \uparrow \infty$, $x_n + y_n \in W_n (n=1, 2, \cdots,)$. This means that $\{\lim (x_n + y_n)\} \ni 0$.

Proof of II$'$. From the hypothesis, we have a sequence $\{U_n\}$ such that

$$U_n \in \mathfrak{V}_{\alpha_n},\ U_n \supseteq U_{n+1},\ \alpha_n \uparrow \infty,\ x_n \in U_n.$$

If $|\lambda| \leq 1$, then $\lambda x_n \in U_n$ (because U_n is circled); therefore, we see at once $\{\lim \lambda x_n\} \ni 0$. Now, suppose $|\lambda| > 1$. Applying axiom (2) to U_1, E, λ, there is a β_1^* and a $V_1^* \in \mathfrak{V}_{\beta_1^*}$ with $\lambda U_1 \subseteq V_1^*$. Since $\lim_{n\to\infty} \psi(\alpha_n, |\lambda|) = \infty$, we can choose an $n_1 > 1$ such that $\psi(\alpha_{n_1}, |\lambda|) > \beta_1^*$. Applying again axiom (2) to U_{n_1}, V_1^*, and λ, there exist a $\beta_2^* \geq \psi(\alpha_{n_1}, |\lambda|)$ and a $V_2^* \in \mathfrak{V}_{\beta_2^*}$ with $\lambda U_{n_1} \subseteq V_2^* \subseteq V_1^*$.

Continuing this process, we obtain sequences $\{n_i\}$, $\{\beta_i^*\}$, $\{V_i^*\}$ such that

$$n_i < n_{i+1},\ \beta_i^* < \beta_{i+1}^*;\ V_i^* \in \mathfrak{V}_{\beta_i^*},\ V_i^* \supseteq V_{i+1}^*,$$
$$\text{and}\quad \lambda x_n \in V_i^* \quad \text{for } n \geq n_{i-1}.$$

Putting $\beta_n = \beta_i^*$, $V_n = V_i^*$ for $n_{i-1} \leq n < n_i (i=1, 2, \cdots,)$ we have $V_n \in \mathfrak{V}_{\beta_n}$, $V_n \supseteq V_{n+1}$, $\beta_n \uparrow \infty$, $\lambda x_n \in V_n$; namely, $\{\lim \lambda x_n\} \ni 0$.

Proof of III′. First, by the axiom (3) (taking E as U, and 1 as n), there is an $\alpha_1 \geq 1$, a $U_1 \in \mathfrak{B}_{\alpha_1}$, and an $\varepsilon_1 > 0$ such that $\varepsilon_1 x \in U_1$.

Next, applying again (3) to U_1, $\varepsilon_1 x$, $\alpha_1 + 1$, we can find an $\alpha_2 > \alpha_1$, $U_2 \in \mathfrak{B}_{\alpha_2}$, and an $\varepsilon_2 > 0$ with $\varepsilon_2 \varepsilon_1 x \in U_2 \subseteq U_1$. Thus, we get sequences $\{\alpha_i\}$, $\{U_i\}$, $\{\varepsilon_i\}$ such that

$$\alpha_i < \alpha_{i+1},\ U \in \mathfrak{B}_{\alpha_i},\ U_i \supseteq U_{i+1}, \quad \text{and} \quad \varepsilon_1 \varepsilon_2 \cdots \varepsilon_i x \in U_i.$$

As $\lim \lambda_n = 0$, we can choose an increasing sequence of integers $\{n_i\}$ which satisfies that $|\lambda_n| \leq \varepsilon_1 \varepsilon_2 \cdots \varepsilon_i$ for $n \geq n_i$. Hence, $\lambda_n x \in U_i$ for $n \geq n_i$.

Put $\beta_n = \alpha_i$, $V_n = U_i$ when $n_i \leq n < n_{i+1} (i = 0, 1, 2, \cdots,)$ where $n_0 = 1$, $\alpha_0 = 0$, $U_0 = E$. Then we have

$$V_n \in \mathfrak{B}_{\beta_n},\ V_n \supseteq V_{n+1},\ \beta_n \uparrow \infty,\ \lambda_n x \in V_n; \text{ that is, } \{\lim \lambda_n x\} \ni 0.$$

This completes our proof.

When the space E satisfies the condition

(*) if $U \in \mathfrak{B}_l$, $V \in \mathfrak{B}_m$, then $U \cap V \in \mathfrak{B}_n$, where $n \geq \max.(l, m)$, axioms (1), (2), (3) can be replaced by simpler ones, (1′), (2′), (3′):[1]

(1′) there exists a function $\phi(\lambda, \mu)$ such as ϕ in (1), and the following holds; for $U \in \mathfrak{B}_l$, $V \in \mathfrak{B}_m$, there is an $n \geq \phi(l, m)$, and a $W \in \mathfrak{B}_n$ such that $U + V \subseteq W$.

(2′) there exists a function $\psi(\lambda, \mu)$ such as ψ in (2), and the following holds; for $U \in \mathfrak{B}_m$, and for a scalar α with $|\alpha| \geq 1$, there is an $n \geq \psi(m, |\alpha|)$ and a $V \in \mathfrak{B}_m$ such that $\alpha U \subseteq V$.

(3′) for any integer n, and for any x in E, there is an $m \geq n$, a $V \in \mathfrak{B}_m$ and a $\rho > 0$ such that $\rho x \in V$.

As is easily seen, (1′), (2′), (3′) are the consequences of (1), (2), (3), respectively. On the other hand, if (*) is satisfied, (1), (2), (3) follow from (1′), (2′), (3′), respectively. For example, suppose (1′), and let $U \in \mathfrak{B}_l$, $V \in \mathfrak{B}_m$, $W \in \mathfrak{B}_n$, $n \leq \phi(l, m)$ and $U + V \subseteq W$. By (1′), there is an $n' \geq \phi(l, m)$ and a W' such that $U + V \subseteq W'$. On account of (*), $W \cap W' \in \mathfrak{B}_{n^*}$, where $n^* \geq \max.(n, n')$, and obviously, $U + V \subseteq W \cap W' \subseteq W$.

Examples. 1. Let Φ be a countably normed space [2]; i.e. a linear space where a sequence of compatible norms $\{\| \ \|_n\}_{n=1,2,\cdots}$ is given, and convergence is defined as convergence with respect to each norm. These norms are assumed monotonously increasing.

Now, let $v(n; 0) = \left\{\varphi \in \Phi \,\middle|\, \|\varphi\|_n < \frac{1}{n}\right\}$ and let $\mathfrak{B}_n$ consist of only one set $v(n; 0)$.[2] Evidently, (A) holds. If $m \geq n$, then $v(n; 0) \supseteq v(m; 0)$ and therefore (*) is satisfied. It is easily verified that (1′), (2′), (3′)

1) Moreover, axiom (B) is the direct consequence of (*), and if none of $\mathfrak{B}_n$ is empty, axiom (a) follows from (*).

2) We put $\mathfrak{B}_0 = \{\Phi\}$. In examples 2, 3, too, we take the whole space as an element of $\mathfrak{B}_0$.

are fulfilled, if we put

$$\phi(\lambda, \mu)=\min.\left(\left[\frac{\lambda}{2}\right], \left[\frac{\mu}{2}\right]\right), \qquad \psi(\lambda, \mu)=\left[\frac{\lambda}{\mu}\right].$$

Thus Φ satisfies all of our axioms.

Convergence of a sequence in Φ as a ranked space is equivalent to the usual convergence; we have $\{\lim \varphi_i\} \ni 0$, if and only if $\|\varphi_i\|_n \to 0$ for every n. In fact, if $\{\lim \varphi_i\} \ni 0$, there is a sequence $\{V_i\}$ such that

$$V_i \in \mathfrak{B}_{\alpha_i},\ V_i \supseteq V_{i+1},\ \alpha_i \uparrow \infty,\ \varphi_i \in V_i.$$

For given n and for given $\varepsilon>0$, we can find some i_0 such that,

$$\text{if } i \geq i_0, \text{ then } \alpha_i \geq n \text{ and } \frac{1}{\alpha_i}<\varepsilon$$

Since $V_{i_0} \in \mathfrak{B}_{\alpha_{i_0}}$, $V=v(\alpha_{i_0}; 0)$. When $i \geq i_0$, $\varphi_i \in V_{i_0}$, consequently, $\|\varphi_i\|_n \leq \|\varphi_i\|_{\alpha_{i_0}} < \dfrac{1}{\alpha_{i_0}} < \varepsilon$. This means that $\|\varphi_i\|_n \to 0$ for every n.

Conversely, suppose that $\|\varphi_i\|_n \to 0$ for any n. Then we can choose a sequence of integers $\{i_n\}$ such that

$$i_n < i_{n+1};\ \|\varphi_i\|_n < \frac{1}{n} \qquad \text{for } i \geq i_n (n=0, 1, 2, \cdots,).$$

Putting $\alpha_i=n$, $V_i=v(n; 0)$, when $i_n \leq i < i_{n+1} (n=0, 1, 2, \cdots,)$, we have

$$V_i \in \mathfrak{B}_{\alpha_i},\ V_i \supseteq V_{i+1},\ \alpha_i \uparrow \infty,\ \varphi_i \in V_i; \quad \text{i.e.} \quad \{\lim \varphi_i\} \ni 0.$$

This completes our proof.

2. L. Schwartz defined the space $\mathcal{D}$ [3], consisting of all infinitely differentiable functions with compact carrier, and convergence in it. Now, let

$$v(n, K; 0)=\left\{\varphi \in \mathcal{D} \mid \text{car. } \varphi \subseteq [-K, K],\ \max_{0 \leq j \leq n-1}.\ \sup_x.\ |\varphi^{(j)}(x)| < \frac{1}{n}\right\}$$

and let $\mathfrak{B}_n$ be the collection of all $v(n, K; 0)$, where K is arbitrary positive number.

Obviously (A) holds. Moreover, it is easily seen that, if $n \leq m$ and $K \leq L$, then $v(m, K; 0) \subseteq v(n, L; 0)$, and that $v(n_1, K_1; 0) \cap v(n_2, K_2; 0) = v(n, K; 0)$, where $n=\max.(n_1, n_2)$, $K=\min.(K_1, K_2)$. Hence $(*)$ holds. Similarly as for Φ, we can see that (1′), (2′), (3′) are also fulfilled, putting μ $\phi(\lambda, \mu)=\min.\left(\left[\dfrac{\lambda}{2}\right], \left[\dfrac{\mu}{2}\right]\right)$ and $\psi(\lambda, \mu)=\left[\dfrac{\lambda}{\mu}\right]$.

The convergence in $\mathcal{D}$ as a ranked space is equivalent to that L. Schwartz defined; we have $\{\lim \varphi_i\} \ni 0$ if and only if there exists some K such that car. $\varphi_i \subseteq [-K, K]$ for every i, and for each fixed n, $\varphi_i^{(n)}(x)$ (and $\varphi_i(x)$ itself) converges to 0 uniformly in $[-K, K]$.

3. Let Φ be a countably normed space, and Φ' be its dual (i.e. linear space consisting of all continuous linear functionals on Φ). It is known that Φ' is the union of Φ'_n where Φ_n is the completion

of Φ with respect to the norm $\|\ \|_n$; in other words, for any f in Φ', there is some p such that $\|f\|'_p<\infty$ (where $\|f\|'_p=\sup_{\|\varphi\|_p\leq 1}|f(\varphi)|$). Moreover, since $\|\ \|_n\leq\|\ \|_{n+1}$, $\|\ \|'_n\geq\|\ \|'_{n+1}$.

Now, let $v(n, p; 0)=\left\{f\in\Phi'_p \mid \|f\|'_p<\frac{1}{n}\right\}$, and let $\mathfrak{B}_n$ be the collection of $v(n, p; 0)$, $p=1, 2, \cdots$. It is clear that, if $n\leq m$ and $p\leq q$, then $v(m, p; 0)\subseteq v(n, q; 0)$. Furthermore, we remark that, if $v(m,p;0)\subseteq v(n, q; 0)$, then necessarily $p\leq q$. In fact, suppose $p>q$. Then $\Phi'_p\supseteq\Phi'_q$. Since we can assume that any two norms $\|\ \|_p$ and $\|\ \|_q$ are not equivalent, and therefore $\|\ \|'_p$ and $\|\ \|'_q$ are not equivalent, we have $\Phi'_p\supsetneqq\Phi'_q$. On the other hand, from $v(m, p; 0)\subseteq v(n, q; 0)$, $\Phi'_p\subseteq\Phi'_q\left(\text{because } \Phi'_p=\bigcup_{m=1}^{\infty} v(m, p; 0) \text{ and } \Phi'_q=\bigcup_{n=1}^{\infty} v(n, q; 0)\right)$. This is a contradiction.

It is easily verified that (A), (B), (a) holds.

Let us show that (1) is satisfied, putting $\phi(\lambda, \mu)=\min.\left(\left[\frac{\lambda}{2}\right], \left[\frac{\mu}{2}\right]\right)$. Let $U=v(l, p; 0)$, $V=v(m, q; 0)$, $W=v(n, r; 0)$, and suppose

$$U+V\subseteq W,\ n\leq\min.\left(\left[\frac{l}{2}\right], \left[\frac{m}{2}\right]\right).$$

Then $U\subseteq W$, $V\subseteq W$, and by the remark above, we have $p\leq r$, $q\leq r$.

Putting $r^*=\max.(p, q)$, $n^*=\min.\left(\left[\frac{l}{2}\right], \left[\frac{m}{2}\right]\right)$, and $W^*=v(n^*, r^*; 0)$, we have $W^*\subseteq W$, because of $n^*\geq n$ and $r^*\leq r$. Moreover, let $f\in U$ and $g\in V$, then

$$\|f+g\|'_{r^*}\leq\|f\|'_{r^*}+\|g\|'_{r^*}\geq\|f\|'_p+\|g\|'_q\leq\frac{1}{l}+\frac{1}{m}\leq\frac{1}{n}$$

Hence $U+V\subseteq W$.

Similarly, we can show that (2) and (3) also hold.

The convergence in Φ' as a ranked space is equivalent to the strong convergence; we have $\{\lim f_i\}\ni 0$ if and only if, there exists some p with $f_i\in\Phi'_p$ for every i, and $\|f_i\|'_p\to 0$. In fact, if $\{\lim f_i\}\ni 0$, there is a sequence $\{V_i\}$, such that

$$V_i\in\mathfrak{B}_{\alpha_i},\ V_i\supseteq V_{i+1},\ \alpha_i\uparrow\infty,\ f_i\in V_i.$$

Let $V_i=v(\alpha_i, p_i; 0)$. From $V_i\supseteq V_{i+1}$, we have $p_i\geq p_{i+1}$. Therefore, for every i, $\|f_i\|'_{p_1}\leq\|f_i\|_{p_i}<\frac{1}{\alpha_i}$. This means that $f_i\in\Phi_{p_1}$, and $\|f_i\|'_{p_1}\to 0$. Conversely, if $\|f_i\|'_p\to 0$ for some p, then, we can show that $\{\lim f_i\}\ni 0$, in the similar way as for the convergence in Φ.

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