126. *On the Representation of Large Even Integers as Sums of a Prime and an Almost Prime. II*[*]

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Recently A. A. Buhštab [3, 4] has given a proof for the following[**]

Theorem. *Every sufficiently large even integer can be represented as a sum of a prime and an almost prime composed of at most three prime factors.*

Here, by an almost prime is meant, in general, an integer>1 with a bounded number of prime factors. His proof of this theorem makes essential use of an important result due to E. Bombieri [1; Theorem 4] (cf. also [7; Theorem 2]) with a complicated combinatorial improvement of the sieve of Eratosthenes, and depends on a long numerical computation for some functions involved therein.

The purpose of the present article is to provide another proof *without* any numerical computation for the theorem stated above.

Almost needless to say, we can also prove that for every fixed integral value of $k\neq 0$ there exist infinitely many primes p such that $p+2k$ has at most three prime factors (cf. [4]).

1. Let k and l be two integers with $k\geqq 1, 0\leqq l<k, (k, l)=1$. Let $\pi(X, k, l)$ denote as usual the number of primes $p\leqq X$ satisfying $p\equiv l \pmod k$. We set

$$R(X, k, l)=\pi(X, k, l)-\frac{\operatorname{li} X}{\phi(k)}$$

and

$$R(X, k)=\max_{(l,k)=1} |R(X, k, l)|,$$

where $\phi(k)$ is the Euler totient function and $\operatorname{li} X$ is the logarithmic integral.

Lemma 1. *For any fixed* $\varepsilon>0$ *and any constant* $A>0$ *we have for* $Y\leqq X^{1/2-\varepsilon}$

$$\sum_{m\leqq Y}\tau(m)R(X, m)=O\left(\frac{X}{\log^A X}\right),$$

where $\tau(n)$ *denotes the number of divisors of* n *and where the* O*-constant may depend on* ε *and* A.

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*) Continuation of the article in Proc. Japan Acad., **40**, 150 (1964).

**) We note that this result was also asserted (without proof) to hold by A. I. Vinogradov [7; Theorem 3] and A. A. Buhštab [4] demonstrated in fact somewhat more.

In respect of the introduction of the additional factor $\tau(m)$, our Lemma 1 is slightly stronger than [7; Theorem 2], which is an immediate consequence of [1; Theorem 4]; however, an inspection of Bombieri's paper [1] will show without difficulty that our lemma holds true in the form presented above.

2. Let ε be a fixed real number with $0<\varepsilon<1/2$ and let N be a sufficiently large even integer. Put $x=N^{1/(2+\varepsilon)}$ and let y and z be two real numbers satisfying $2<y, z\leqq x$. We set

$$P_z=P_{z,N}=\prod_{\substack{p<z\\ p\nmid N}} p.$$

For any positive integer d we define $S_d(z)$ to be the number of primes $p\leqq N$ such that $p\equiv N \pmod d$ and $(N-p, P_z)=1$.

Suppose now that $(d, NP_z)=1$. Then, if $m\mid P_z$, we have $(m, dN)=1$ and the number of primes $p\leqq N$ satisfying $N-p\equiv 0 \pmod{dm}$ is equal to $\pi(N, dm, a)$ for a suitable a with $(a, dm)=1$. Hence:

Lemma 2. *Put $f_1(m)=\sum_{n\mid m}\mu(n)\phi(m/n)$. If $(d, NP_z)=1$, then we have for any y with $2<y\leqq x$*

$$S_d(z)\leqq\frac{\operatorname{li} N}{\phi(d)}\Big(\sum_{\substack{m\leqq y/d\\ m\mid P_z}}\frac{1}{f_1(m)}\Big)^{-1}+R$$

with

$$R\leqq\sum_{\substack{m_1,m_2\leqq y/d\\ m_1,m_2\mid P_z}}|\lambda_{m_1}\lambda_{m_2}R(N, d[m_1, m_2])|,$$

where $[m_1, m_2]$ denotes the least common multiple of m_1, m_2, and

$$\lambda_m=\mu(m)\frac{\phi(m)}{f_1(m)}\Big(\sum_{\substack{n\leqq y/m\\ n\mid P_z}}\frac{1}{f_1(n)}\Big)\Big(\sum_{\substack{n\leqq y\\ n\mid P_z}}\frac{1}{f_1(n)}\Big)^{-1}.$$

This is a well-known upper estimate in the sieve of A. Selberg (cf. [5; §3]).

By a general theorem of N. G. de Bruijn and J. H. van Lint [2] we have, using a result of J. H. van Lint and H.-E. Richert [6],

$$\sum_{\substack{m\leqq y/d\\ m\mid P_z}}\frac{1}{f_1(m)}=\theta(v)\sum_{\substack{m\leqq z\\ (m,N)=1}}\frac{\mu^2(m)}{f_1(m)}+O(1)$$

uniformly for $0<v_0\leqq v<\infty$, where $v=(\log(y/d))/\log z$ and where the O-constant is uniform in N. Here $\theta(v)=\theta_1(v)$ is the function of v defined in [2]. In particular, we have $\theta(v)=v$ for $0\leqq v\leqq 1$ and it is shown in [2, 6] that $\theta(v)$ is a strictly increasing function of v and that $\theta(v)=e^C+O(e^{-v})(v\geqq 0)$, C being the Euler constant.

Now, one may easily verify that

$$\sum_{\substack{m\leqq z\\ (m,N)=1}}\frac{\mu^2(m)}{f_1(m)}=\frac{\phi(N)}{N}\prod_{p\nmid N}\Big(1+\frac{1}{p(p-2)}\Big)\log z+O(\log\log N).$$

It thus follows from Lemma 2 with $y=(dx)^{1/2}$ that

$$S_d(z) \leqq \frac{\operatorname{li} N}{\phi(d)} P(z)\left(\frac{e^C}{\theta(u/2)} + O\left(\frac{(\log\log N)^2}{\log z}\right)\right) + R$$

uniformly for d satisfying $(d, NP_z)=1, 1 \leqq d \leqq x^\alpha (0<\alpha<1)$, where $u=(\log(x/d))/\log z$ and

$$P(z)=P_N(z)=\prod_{\substack{p<z \\ p\nmid N}}\left(1-\frac{1}{p-1}\right).$$

This last inequality is effective, however, only for z not too small, that is, only for $z>z_0=\exp\log^\beta x$ $(0<\beta<1)$, say. For $2<z\leqq z_0$ we may use the sieve method of V. Brun instead (cf. [4]) to obtain

$$S_d(z) \leqq \frac{\operatorname{li} N}{\phi(d)} P(z)\left(1+O\left(\frac{1}{\log x}\right)\right)+R,$$

which is again valid uniformly for d such that $(d, NP_z)=1, 1\leqq d\leqq x^\alpha$ $(0<\alpha<1)$.

3. Let $\psi_i(u)$ $(i=1, 2)$ be the functions defined for all real u, and satisfying the following conditions: for $i=1, 2$

(i) $\psi_i(u)$ is continuous for $u>0$,

(ii) $\psi_i(u)=0$ for $u<0$,

(iii) $\psi_i(u)=1/u$ for $0<u\leqq 2$,

(iv) $u\psi_i'(u)=-\psi_i(u)+(-1)^i\psi_i(u-1)$ for $u>2$.

Obviously $\psi_i(u)$ $(i=1, 2)$ are uniquely determined by these conditions. With these two functions we set for real u

$$G(u)=e^C(\psi_2(u)+\psi_1(u))$$

and

$$g(u)=e^C(\psi_2(u)-\psi_1(u)).$$

It is not difficult to see that we have

$$G(u)=\frac{2e^C}{u}(0<u\leqq 3),\ g(u)=\begin{cases}0 & (0<u\leqq 2),\\ \dfrac{2e^C\log(u-1)}{u} & (2<u\leqq 4),\end{cases}$$

$$(uG(u))'=g(u-1)\ (u>2),\ (ug(u))'=G(u-1)\ (u>2),$$

and that $G(u)$ and $g(u)$ are respectively monotonically decreasing and monotonically increasing functions of $u>0$ such that

$$G(u)=1+O(e^{-u})\ (u\geqq 1),\ g(u)=1+O(e^{-u})\ (u\geqq 1)$$

(see [5; § 5]).

Now suppose again that $(d, NP_z)=1$. We have then

$$S_d(z)=\pi(N, d, a)-\sum_{\substack{p<z \\ p\nmid N}} S_{dp}(p)$$

for a suitable a with $(a, d)=1$. Using the identity

$$P(z)=1-\sum_{\substack{p<z \\ p\nmid N}}\frac{P(p)}{p-1}$$

and the results of § 2 above and following *mutatis mutandis* the lines of arguments in [5], we can prove:

Lemma 3. *If* $2<z\leqq x$ *and* $(d, NP_z)=1, 1\leqq d\leqq x^\alpha$ $(0<\alpha<1)$, *then* *we have for some constants $B>0$ and $c>0$*

$$S_d(z)\leqq \frac{\mathrm{li}\,N}{\phi(d)}P(z)\Big(G(u)+O\Big(\frac{(\log\log N)^2}{\log^B x}\Big)\Big)+R_d$$

and

$$S_d(z)\geqq \frac{\mathrm{li}\,N}{\phi(d)}P(z)\Big(g(u)+O\Big(\frac{(\log\log N)^2}{\log^B x}\Big)\Big)-R_d$$

with

$$R_d=O((\log^C x)\,|\,R\,|),$$

where $u=(\log(x/d))/\log z$ and the constants implied by the symbol O are uniform in d.

4. We now put

$$z_1=x^{1/(3-2\varepsilon)},\ z=x^{2/(3-2\varepsilon)},$$

where $x=N^{1/(2+\varepsilon)}$ $(0<\varepsilon<1/2)$. Using Lemma 1 and Lemma 3 with $d=1$, we find

$$S_1(z_1)\geqq \mathrm{li}\,N\cdot P(z_1)(g(3-2\varepsilon)+o(1))$$

since $(\log x)/\log z_1=3-2\varepsilon$. Also, by Lemmas 1 and 3 again, we have, writing u_q for $(\log(x/q))/\log z_1$ for any prime q,

$$\sum_{\substack{z_1\leqq q<z\\ q\nmid N}} S_q(z_1)\leqq \mathrm{li}\,N\cdot P(z_1)\sum_{\substack{z_1\leqq q<z\\ q\nmid N}}\frac{1}{q-1}(G(u_q)+o(1))$$
$$\leqq \mathrm{li}\,N\cdot P(z_1)\Big(\int_{(3-2\varepsilon)/2}^{3-2\varepsilon} G\Big((3-2\varepsilon)\Big(1-\frac{1}{t}\Big)\Big)\frac{dt}{t}+o(1)\Big).$$

Let S be the number of primes $p\leqq N$ such that $N-p$ is not divisible by any prime $q<z_1$, $(q,N)=1$, divisible by at most two distinct primes q with $z_1\leqq q<z$, $(q,N)=1$, and not divisible by any integer of the form q^2 with $z_1\leqq q<z$, $(q,N)=1$. Then it is clear that

$$S\geqq S_1(z_1)-\frac{1}{3}\sum_{\substack{z_1\leqq q<z\\ q\nmid N}} S_q(z_1)+O\Big(\frac{N}{z_1}\Big)+O(z)$$
$$\geqq \mathrm{li}\,N\cdot P(z_1)(K_\varepsilon+o(1)),$$

where

$$K_\varepsilon=g(3-2\varepsilon)-\frac{1}{3}\int_{(3-2\varepsilon)/2}^{3-2\varepsilon} G\Big((3-2\varepsilon)\Big(1-\frac{1}{t}\Big)\Big)\frac{dt}{t}$$

is a continuous function of ε $(0\leqq\varepsilon<1/2)$.

Now we find $K_0=2e^C(\log 2)/9$. Hence, we must have $K_\varepsilon>e^C(\log 2)/9>0$ for some sufficiently small value of $\varepsilon(0<\varepsilon<1/2)$. It follows that for such ε we have $S>2$ for all large enough even N. Since $z_1>N^{1/6}$, $z>N^{1/3}$, and $N=p+(N-p)$, this completes the proof of the theorem.

Note added in proof (September 23, 1967). The theorem has also been proved in like manner by H. Halberstam, W. Jurkat, and H.-E. Richert, Un nouveau résultat de la méthode du crible, C. R. Acad. Sci. Paris, t. 264, 920–923 (1967).

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