167. On Closed Mappings and M-Spaces. II

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1. Introduction. The main purpose of this paper is to give the affirmative answer to an open problem raised by A. Arhangel'skii in his recent communication to K. Morita whether the image Y under a perfect mapping f of a paracompact normal M-space X is an M-space or not.[1] A closed continuous mapping f of a topological space X onto a topological space Y is said to be perfect if the inverse images under f of points y of Y are compact subspaces of X. We shall prove the following main theorem.

Theorem 1.1. *Let f be a closed continuous mapping of an M-space X onto a normal space Y, where X is T_1. If $f^{-1}(y)$ is countably compact for any point y of Y, then Y is also an M-space.*

As a direct consequence of Theorem 1.1 we obtain the following

Cororally 1.2. *Let f be a closed continuous mapping of a normal M-space X onto a topological space Y, where X is T_1. If $f^{-1}(y)$ is countably compact for any point y of Y, then Y is also a normal M-space.*

Some applications and a generalization of our main theorem will be mentioned in §4.

2. Lemmas. Lemma 2.1. *Let T be a metric space. If $\{\mathfrak{F}_n\}$ is a sequeuce of locally finite closed coverings of T such that $\{\mathfrak{F}_n\}$ satisfies the condition $(*)$ and that $\mathfrak{F}_{n+1}$ is a refinement of $\mathfrak{F}_n$ for every n, then there exists a sequence $\{\mathfrak{U}_{nm} \mid n=1, 2, \cdots; m=1, 2, \cdots\}$ of locally finite open coverings of T such that*

(1) $\{\mathfrak{U}_{nm}\}$ satisfies the condition $(*)$,

(2) $F_{n\lambda} \subset U_{nm\lambda}$ for $\lambda \in \Lambda_n$; $n=1, 2, \cdots$, $m=1, 2, \cdots$,

where $\mathfrak{F}_n=\{F_{n\lambda} \mid \lambda \in \Lambda_n\}$ and $\mathfrak{U}_{nm}=\{U_{nm\lambda} \mid \lambda \in \Lambda_n\}$.

Proof. For any $F_{n\lambda}$ of $\mathfrak{F}_n$, let us put

$$V_{nm\lambda}=\{x \mid d(x, F_{n\lambda})<1/m\},$$

where d is a metric function in T and m is an arbitrary positive integer. Clearly $F_{n\lambda} \subset V_{nm\lambda}$. Let us put further

$$\mathfrak{B}_{nm}=\{V_{nm\lambda} \mid \lambda \in \Lambda_n\}.$$

Then we can prove that $\{\mathfrak{B}_{nm}\}$ satisfies the condition $(*)$. Indeed, let $\mathfrak{K}^k=\{K_i \mid i=1, 2, \cdots\}$ be a family of subsets of T which has the finite intersection property and contains as a member a subset of

1) Prof. K. Morita has kindly informed me of this open problem.

$\mathrm{St}(x_0, \mathfrak{V}_{nm})$ for every n, m and for some fixed point x_0 of T. We can assume without loss of generality that $K_{i+1} \subset K_i$ for every i. Let $K_{i(n,m)} \subset \mathrm{St}(x_0, \mathfrak{V}_{nm})$ for any n, m, and let us put

$$\varepsilon_n(x_0) = d(x_0, \cup)\{F \mid x_0 \notin F, F \in \mathfrak{F}_n\}$$

for each n. Then clearly $\varepsilon_n(x_0) > 0$. Further, if $1/m < \varepsilon_n(x_0)$, then $\mathrm{St}(x_0, \mathfrak{V}_{nm}) = S(\mathrm{St}(x_0, \mathfrak{F}_n); 1/m)$, and hence

$$S(K_{i(n,m)}; 1/\mathrm{m}) \cap \mathrm{St}(x_0, \mathfrak{F}_n) \neq \phi,$$

where $S(A; \varepsilon) = \{x \mid d(x, A) < \varepsilon\}$ for any subset A of T and for any $\varepsilon > 0$. Consequently for each n we can find a positive integer m_n and a point x_n of T such that (1) $1/m_n < \varepsilon_n(x_0)$, $n < m_n$, (2) $i(n, m_n) > n$, and (3) $x_n \in S(K_{i(n,m_n)}; 1/m_n) \cap \mathrm{St}(x_0, \mathfrak{F}_n)$. If we put $A_k = \{x_n \mid n \geqq k\}$, then by the condition (*) for $\{\mathfrak{F}_n\}$ we have

$$\cap \{\bar{A}_k \mid k = 1, 2, \cdots\} \neq \phi.$$

Let $t_0 \in \cap \{\bar{A}_k \mid k = 1, 2, \cdots\}$. Then it can be proved that

$$t_0 \in \cap \{\bar{K}_i \mid i = 1, 2, \cdots\}.$$

If otherwise, then there exists some $\varepsilon > 0$ and some positive integer i_0 such that

$$S(t_0; \varepsilon) \cap K_j = \phi \qquad \text{for any} \quad j \geqq i_0.$$

Let n be a positive integer such that $3/\varepsilon < n$, $i_0 < n$ and $d(t_0, x_n) < \varepsilon/3$. Then there exists a point y_n of $K_{i(n,mn)}$ such that

$$d(x_n, y_n) < 1/m_n < 1/n < \varepsilon/3.$$

Since $d(t_0, y_n) < 2\varepsilon/3 < \varepsilon$, we have

$$S(t_0, \varepsilon) \cap K_{i(n,m_n)} \neq \phi.$$

This is a contradiction, because $i(n, m_n) > n > i_0$. Thus $\{\mathfrak{V}_{nm}\}$ satisfies the condition (*).

Finally for each n we can find a locally finite open covering $\mathfrak{W}_n = \{W_{n\lambda} \mid \lambda \in \Lambda_n\}$ of T such that $F_{n\lambda} \subset W_{n\lambda}$ for any $\lambda \in \Lambda_n$. This is possible in case Y is strongly normal, i.e., collectionwise normal and countably paracompact (cf. M. Katětov [2]). Let us put $U_{nm\lambda} = V_{nm\lambda} \cap W_{n\lambda}$, $\mathfrak{U}_{nm} = \{U_{nm\lambda} \mid \lambda \in \Lambda_n\}$. Then each $\mathfrak{U}_{nm}$ is a locally finite open covering of T, and $\{\mathfrak{U}_{nm}\}$ satisfies the conditions (1) and (2). Thus we complete the proof.

Lemma 2.2. *Let Y be a topological space in which there exists a sequence $\{\mathfrak{V}_n\}$ of (not necessarily open or closed) coverings of Y satisfying the condition (*), and f a closed continuous mapping of a topological space X onto Y. If $f^{-1}(y)$ is countably compact for any point y of Y, then $\{\mathfrak{U}_n\}$ satisfies also the condition (*), where $\mathfrak{U}_n = f^{-1}(\mathfrak{V}_n)$.*

Since this lemma can be proved similarly as [1, Theorem 2.4], we omit the proof.

3. Proof of Theorem 1.1. Let $\{\mathfrak{U}_n\}$ be a normal sequence of open coverings of X which satisfies the condition (*). Then there exists a normal sequence $\{\mathfrak{V}_n\}$ of locally finite open coverings of X

such that $\overline{\mathfrak{V}}_n$ is a refinement of $\mathfrak{U}_n$, where $\overline{\mathfrak{V}}_n=\{\bar{V} \mid V\in\mathfrak{V}_n\}$. For brevity we put $\mathfrak{F}_n=\overline{\mathfrak{V}}_n$ for every n. Then $\mathfrak{F}_n$ is a locally finite closed covering of X and $\mathfrak{F}_{n+1}$ is a refinement of $\mathfrak{F}_n$ for each n. Furthermore it is clear that $\{\mathfrak{F}_n\}$ satisfies the condition $(*)$. Let us put $\mathfrak{F}_n=\{F_{n\lambda} \mid \lambda\in\Lambda_n\}$, $L_{n\lambda}=f(F_{n\lambda})$, $\mathfrak{L}_n=\{L_{n\lambda} \mid \lambda\in\Lambda_n\}$. Then $\mathfrak{L}_{n+1}$ is a refinement of $\mathfrak{L}_n$ for every n, and by the proof of [1, Theorem 2.3] $\{\mathfrak{L}_n\}$ is a sequence of locally finite closed coverings of Y which satisfies the condition $(*)$. If we put $M_{n\lambda}=f^{-1}(L_{n\lambda})$, $\mathfrak{M}_n=\{M_{n\lambda} \mid \lambda\in\Lambda_n\}$, then $\mathfrak{M}_{n+1}$ is a refinement of $\mathfrak{M}_n$ for every n, and by the proof of [1, Theorem 2.4] $\{\mathfrak{M}_n\}$ is a sequence of locally finite closed coverings of X which satisfies the condition $(*)$. We note that $F_{n\lambda}\subset M_{n\lambda}$.

Now, since X is an M-space, there exists a closed continuous mapping g of X onto a metrizable space T such that $g^{-1}(t)$ is countably compact for any point t of T (cf. [4, Theorem 6.1]). Let us put $S_{n\lambda}=g(M_{n\lambda})$, $\mathfrak{S}_n=\{S_{n\lambda} \mid \lambda\in\Lambda_n\}$. Then $\mathfrak{S}_{n+1}$ is a refinement of $\mathfrak{S}_n$ for every n, and by the proof of [1, Theorem 2.3] $\{\mathfrak{S}_n\}$ is a sequence of locally finite closed coverings of T which satisfies the condition $(*)$. Hence by Lemma 2.1 there exists a sequence $\{\mathfrak{O}_{nm}\}$ of locally finite open coverings of T such that

(1) $\{\mathfrak{O}_{nm}\}$ satisfies the condition $(*)$,

(2) $S_{n\lambda}\subset O_{nm\lambda}$,

where $\mathfrak{O}_{nm}=\{O_{nm\lambda} \mid \lambda\in\Lambda_n\}$. If we put further $W_{nm\lambda}=g^{-1}(O_{nm\lambda})$, $\mathfrak{W}_{nm}=\{W_{nm\lambda} \mid \lambda\in\Lambda_n\}$, then $M_{n\lambda}\subset W_{nm\lambda}$ for each n, m, and λ, and by the proof of [1,T heorem 2.4] $\{\mathfrak{W}_{nm}\}$ is a sequence of locally finite open coverings of X which satisfies the condition $(*)$. Let us put

$$G_{nm\lambda}=Y-f(X-W_{nm\lambda}).$$

Since f is a closed mapping of X onto Y, each $G_{nm\lambda}$ is open in Y, and $L_{n\lambda}\subset G_{nm\lambda}$, $M_{n\lambda}\subset f^{-1}(G_{nm\lambda})\subset W_{nm\lambda}$. Finally let us put

$$\mathfrak{G}_{nm}=\{G_{nm\lambda} \mid \lambda\in\Lambda_n\}$$

for each n, m. Then each $\mathfrak{G}_{nm}$ is a locally finite open covering of Y. This follows from [1, Lemma 2.1], because $\{f^{-1}(G_{nm\lambda}) \mid \lambda\in\Lambda_n\}$ is locally finite in X. Furthermore it can be proved that $\{\mathfrak{G}_{nm}\}$ satisfies the condition $(*)$. In fact, let $\mathfrak{K}$ be a family consisting of a countable number of subsets of Y which has the finite intersection property and contains as a member a subset of $\mathrm{St}\,(y_0, \mathfrak{G}_{nm})$ for every n, m, and for some point y_0 of Y. If we put $\mathfrak{K}^*=\{f^{-1}(K) \mid K\in\mathfrak{K}\}$, then $\mathfrak{K}^*$ is a family consisting of a countable number of subsets of X which has the finite intersection property, and further contains as a member a subset of $\mathrm{St}\,(x_0, \mathfrak{W}_{nm})$ for every n, m, where x_0 is an arbitrary point of $f^{-1}(y_0)$. Consequently we have $\cap\{\overline{f^{-1}(K)} \mid K\in\mathfrak{K}\}\neq\phi$, which implies that $\cap\{\bar{K} \mid K\in\mathfrak{K}\}\neq\phi$. Thus $\{\mathfrak{G}_{nm}\}$ satisfies the condition $(*)$. By a suitable ordering of $\{\mathfrak{G}_{nm}\}$ we can put $\{\mathfrak{G}_{nm}\}=\{\mathfrak{G}_n \mid n=1, 2, \cdots\}$.

Since Y is a normal space, any locally finite open covering of Y is normal (cf. A. H. Stone [7]). Hence there exists a normal sequence $\{\mathfrak{H}_n\}$ of open coverings of Y such that $\mathfrak{H}_n$ is a refinement of $\mathfrak{G}_n$ for each n. It is obvious that $\{\mathfrak{H}_n\}$ satisfies the condition (*). Thus we complete the proof.

4. Applications and a generalization of the main theorem.

Theorem 4.1. *Let Y be the image under a closed continuous mapping f of a normal M-space X, where X is T_1. Then the following statements are equivalent.*

(1) *Y is an M-space.*

(2) *Y is a q-space in the sense of E. Michael [3].*

(3) *The boundary $\mathfrak{B}f^{-1}(y)$ of the inverse image $f^{-1}(y)$ is countably compact for every point y of Y.*

Proof. The implication (1)→(2) is trivial, and (2)→(3) was proved by E. Michael [3]. Hence it is sufficient to prove only (3)→(1). For each point y of Y, we shall define an open subset $L(y)$ of X as follows:

$$L(y)=\begin{cases}\operatorname{Int} f^{-1}(y), & \text{if } \mathfrak{B}f^{-1}(y)\neq\phi,\\ f^{-1}(y)-p_y, & \text{if } \mathfrak{B}f^{-1}(y)=\phi,\end{cases}$$

Where p_y is an arbitrary point of $f^{-1}(y)$ (cf. [5]). Let us put

$$L=\cup\{L(y)\mid y\in Y\},\qquad F=X-L.$$

Then F is a closed subset of X. Since any closed subspace of an M-space is also an M-space, F is an M-space as a subspace of X. If we denote by $\tilde{f}$ the restriction of f on F, then the mapping $\tilde{f}\colon F\to Y$ is closed, continuous and $\tilde{f}^{-1}(y)$ is countably compact for any point y of Y. Hence by Theorem 1.1, Y is an M-space. Thus we complete the proof.

Theorem 4.2. (K. Morita and S. Hanai [5, Theorem 1]). *Let f be a closed continuous mapping of a metric space X onto a topological space Y. In order that Y be metrizable it is necessary and sufficient that the boundary $\mathfrak{B}f^{-1}(y)$ of the inverse image $f^{-1}(y)$ be compact for every point y of Y.*

Proof. If Y is metrizable, then it is an M-space. Hence by Theorem 4.1, the boundary $\mathfrak{B}f^{-1}(y)$ is compact for every point y of Y. To prove sufficiency, it suffices to consider the case when f is perfect, i.e., $f^{-1}(y)$ is compact for every point y of Y. As is well known, the image under a closed continuous mapping of a paracompact Hausdorff space is also a paracompact Hausdorff space. Hence by Theorem 1.1, Y is a paracompact Hausdorff M-space. Since the product mapping $f\times f\colon X\times X\to Y\times Y$ is perfect, the product space $Y\times Y$ is perfectly normal as the image under a closed continuous mapping $f\times f$ of a perfectly normal space $X\times X$. Therefore by a metrization theorem of Okuyama [6], Y is metrizable. Thus we complete the proof.

Now let $\mathfrak{m}$ be an infinite cardinal. We shall say that a topological space X is an $M(\mathfrak{m})$-space if there exists a normal sequence $\{\mathfrak{U}_i\}$ of open coverings of X satisfying the condition below:

$$(**)\begin{cases}\text{If a family } \mathfrak{K} \text{ consisting of at most } \mathfrak{m} \text{ subsets of } X \text{ has the} \\ \text{finite intersection property and contains as a member a subset} \\ \text{of } \mathrm{St}(x_0, \mathfrak{U}_i) \text{ for every } i \text{ and for some fixed point } x_0 \text{ of } X, \text{ then} \\ \cap\{\bar{K} \mid K \in \mathfrak{K}\} \neq \phi.\end{cases}$$

In case $\mathfrak{m}=\aleph_0$, $M(\aleph_0)$-spaces are M-spaces.

As for $M(\mathfrak{m})$-spaces, we can prove analogously the following theorems.

Theorem 4.3. *A topological space X is an $M(\mathfrak{m})$-space if and only if there exists a closed continuous mapping f of X onto a metrizable space T such that $f^{-1}(t)$ is $\mathfrak{m}$-compact for each point t of T.*

Theorem 4.4. *Let f be a closed continuous mapping of an $M(\mathfrak{m})$-space X onto a normal space Y, where X is T_1. If $f^{-1}(y)$ is $\mathfrak{m}$-compact for any point y of Y,* then *Y is also an $M(\mathfrak{m})$-space.*

Corollary 4.5. *Let f be a closed continuous mapping of a normal $M(\mathfrak{m})$-space X onto a topological space Y, where X is T_1. If $f^{-1}(y)$ is $\mathfrak{m}$-compact for any point y of Y, then Y is also a normal $M(\mathfrak{m})$-space.*

References

[1] T. Ishii: On closed mappings and M-spaces. I. Proc. Japan Acad, **43**, 752–756 (1967).

[2] M. Katětov: Extensions of locally finite coverings. Colloq. Math., **6**, 145–151 (1958).

[3] E. Michael: A note on closed maps and compact sets. Israel J. Math., **2**, 173–176 (1964).

[4] K. Morita: Products of normal spaces with metric spaces. Math. Ann., **154**, 365–382 (1964).

[5] K. Morita and S. Hanai: Closed mappings and metric spaces. Proc. Japan Acad., **32**, 10–14 (1956).

[6] A. Okuyama: On metrization of M-spaces. Proc. Japan Acad., **40**, 176–179 (1964).

[7] A. H. Stone: Paracompactness and product spaces. Bull. Amer. Math. Soc., **54**, 977–982 (1948).