164. *On Equivalences of Laws in Elementary Protothetics. I*

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In his paper [1], J. Słupecki has given some generalizations of the six laws that have described the properties of functions of one argument in elementary protothetics.

In this paper, by using the well known rules of inference and substitution we shall show that each laws on functions of one argument is equivalent to its corresponding laws of functions of two arguments. J. Słupecki has not given the proofs of the equivalences given below in his parper [1]. The rules of inference and of substitution used in the systems of elementary protothetics has given in J. Słupecki [1] in detail.

First of all, we shall prove the equivalence of the theorems (a) and (a′) which have been called the *law of development*:

(a) $[f, p]\{f(p)\equiv(f(1)\cdot p\vee f(0)\cdot\sim(p))\}$,

(a′) $[f, p, q]\{f(p, q)\equiv(f(1,1)\cdot p\cdot q\vee f(1,0)\cdot p\cdot\sim(q)\vee f(0,1)\cdot\sim(p)\cdot q\vee f(0,0)\cdot\sim(p)\cdot\sim(q))\}$.

Then we shall first prove the following theorem to show the equivalence mentioned above.

Theorem 1. $[f, p, q]\{[f, r]\{f(r)\supset(f(1)\cdot r\vee f(0)\cdot\sim(r))\}\cdot f(p, q)\supset(f(1, 1)\cdot p\cdot q\vee f(1, 0)\cdot p\cdot\sim(q)\vee f(0, 1)\cdot\sim(p)\cdot q\vee f(0,0)\cdot\sim(p)\cdot\sim(q))\}$.

Proof. The idea of the proof is due to J. Słupecki ([1], p. 71).

(1) $[f, r]\{f(r)\supset(f(1)\cdot r\vee f(0)\cdot\sim(r))\}$,

(2) $f(p, q)\supset$

D1 $[f, p, q]\{\psi\langle f, p\rangle(q)\equiv f(q, p)\}$.

By replacing a functor $\psi\langle f, q\rangle$ for the functor f, a variable p for the variable r in the expression (1), we obtain the following expression.

(3) $\psi\langle f, q\rangle(p)\supset(\psi\langle f, q\rangle(1)\cdot p\vee\psi\langle f, q\rangle(0)\cdot\sim(p))$, (D1; 1)

(4) $\psi\langle f, q\rangle(p)$, (D1; 2)

we obtain the following expression by applying the rule of detachment from (3) and (4).

(5) $\psi\langle f, q\rangle(1)\cdot p\vee\psi\langle f, q\rangle(0)\cdot\sim(p)$, (3; 4)

(6) $f(1, q)\cdot p\vee f(0, q)\cdot\sim(p)$, (D1; 5)

(7) $\chi\langle f, 1\rangle(q)\equiv(\chi\langle f, 1\rangle(1)\cdot q\vee\chi\langle f, 1\rangle(0)\cdot\sim(q))$. (a)

To obtain (7) we have used the functor obtained by the definition D2 given below:

D2 $[f, p, q]\{\chi\langle f, p\rangle(q)\equiv f(p, q)\}$,

(8) $f(1, q)\equiv(f(1, 1)\cdot q\vee f(1, 0)\cdot\sim(q))$, (D2; 7)

(9) $f(1, q)\cdot p\equiv(f(1, 1)\cdot p\cdot q\vee f(1, 0)\cdot p\cdot\sim(q))$, (8)

in obtaining the expression (9) we based ourselves on the following theorem of the propositional calculus:

$$[p, q, r, s]\{(q\equiv(r\vee s))\supset(p\cdot q\equiv(p\cdot r\vee p\cdot s))\}.$$

(10) $\chi\langle f, 0\rangle(q)\equiv(\chi\langle f, 0\rangle(1)\cdot q\vee\chi\langle f, 0\rangle(0)\cdot\sim(q))$, (a)

(11) $f(0, q)\equiv(f(0, 1)\cdot q\vee f(0, 0)\cdot\sim(q))$, (D2; 10)

(12) $f(0, q)\cdot\sim(p)\equiv(f(0, 1)\cdot\sim(p)\cdot q\vee f(0, 0)\cdot\sim(p)\cdot\sim(q))$, (11)

$f(1, 1)\cdot p\cdot q\vee f(1, 0)\cdot p\cdot\sim(q)\vee f(0,1)\cdot\sim(p)\cdot q$
$\vee f(0, 0)\cdot\sim(p)\cdot\sim(q)$. (6; 9; 12)

To obtain the consequent we have used the following theorem of the propositional calculus:

$$[p, p', q, q']\{(p\vee q)\cdot(p\equiv p')\cdot(q\equiv q')\supset(p'\vee q')\},$$

therefore we complete the proof of Theorem 1.

Theorem 2. $[f,p]\{[f,r,s]\{f(r,s)\supset(f(1,1)\cdot r\cdot s\vee f(1,0)\cdot r\cdot\sim(s)\vee f(0, 1)\cdot\sim(r)\cdot s\vee f(0, 0)\cdot\sim(r)\cdot\sim(s))\}\cdot f(p)\supset(f(1)\cdot p\vee f(0)\cdot\sim(p))\}$.

Proof. (1) $[f, r, s]\{f(r, s)\supset(f(1, 1)\cdot r\cdot s\vee f(1, 0)\cdot r\cdot\sim(s)\vee f(0, 1)\cdot\sim(r)\cdot s\vee f(0, 0)\cdot\sim(r)\cdot\sim(s)\}$,

(2) $f(p)\supset$

(3) $\chi\langle f, 1\rangle(q)\equiv(\chi\langle f, 1\rangle(1)\cdot q\vee\chi\langle f, 1\rangle(0)\cdot\sim(q))$, (a)

(4) $f(1, q)\equiv(f(1, 1)\cdot q\vee f(1, 0)\cdot\sim(q))$, (D2; 3)

(5) $f(1, q)\cdot p\equiv(f(1, 1)\cdot p\cdot q\vee f(1, 0)\cdot p\cdot\sim(q))$, (4)

(6) $\chi\langle f, 0\rangle(q)\equiv(\chi\langle f, 0\rangle(1)\cdot q\vee\chi\langle f, 0\rangle(0)\cdot\sim(q))$, (a)

(7) $f(0, q)\equiv(f(0, 1)\cdot q\vee f(0, 0)\cdot\sim(q))$, (D2; 6)

(8) $f(0, q)\cdot\sim(p)\equiv(f(0, 1)\cdot\sim(\mathrm{p})\cdot q\vee f(0, 0)\cdot\sim(p)\cdot\sim(q))$, (7)

(9) $[f, p, q]\{f(p, q)\supset(f(1, q)\cdot p\vee f(0, q)\cdot\sim(p))\}$. (1; 5; 8)

Next we introduce the following definition:

$$[p]\{vr(p)\equiv(p\equiv p)\},$$
$$[f, p, q]\{\omega\Leftarrow f\Rightarrow(p, q)\equiv(f(p)\equiv vr(q))\}.$$

(10) $\omega\Leftarrow f\Rightarrow(p, q)\supset(\omega\Leftarrow f\Rightarrow(1, q)\cdot p\vee\omega\Leftarrow f\Rightarrow(0, q)\cdot\sim(p))$. (9)

By applying the theorem

$$[f, p, q]\{\omega\Leftarrow f\Rightarrow(p, q)\equiv f(p)\}$$

in elementary protothetics for the expression (10), we obtain the following expression:

(11) $f(p)\supset(f(1)\cdot p\vee f(0)\cdot\sim(p))$. (10)

By the rule of detachment,

$$f(1)\cdot p\vee f(0)\cdot\sim(p) \quad (2)$$

we have proved Theorem 2.

Therefore we have obtained Theorem 3 by applying the following theorem of the propositional calculus to Theorems 1 and 2:

$$[p, q, r, s]\{((p\supset q)\cdot r\supset s)\supset(((r\supset s)\cdot p\supset q)\supset((p\supset q)\equiv(r\supset s)))\}.$$

Theorem 3. $[f,p]\{f(p)\supset(f(1)\cdot p\vee f(0)\cdot\sim(p))\}\equiv[f,p,q]\{f(p,q)\supset(f(1,1)\cdot p\cdot q\vee f(1,0)\cdot p\cdot\sim(q)\vee f(0,1)\cdot\sim(p)\cdot q\vee f(0,0)\cdot\sim(p)\cdot\sim(q))\}$.

We shall now prove the converse of the implication.

Theorem 4. $[f,p,q]\{[f,r]\{(f(1)\cdot r\vee f(0)\cdot\sim(r))\supset f(r)\}\cdot(f(1,1)\cdot p\cdot q\vee f(1,0)\cdot p\cdot\sim(q)\vee f(0,1)\cdot\sim(p)\cdot q\vee f(0,0)\cdot\sim(p)\cdot\sim(q))\supset f(p,q)\}$.

Proof. (1) $[f,r]\{(f(1)\cdot r\vee f(0)\cdot\sim(r))\supset f(r)\}$,

(2) $(f(1,1)\cdot p\cdot q\vee f(1,0)\cdot p\cdot\sim(q)\vee f(0,1)\cdot\sim(p)\cdot q\vee f(0,0)\cdot\sim(p)\cdot\sim(q))\supset$

(3) $\chi<f,1>(q)\equiv(\chi<f,1>(1)\cdot q\vee\chi<f,1>(0)\cdot\sim(q))$, (a)

(4) $f(1,q)\equiv(f(1,1)\cdot q\vee f(1,0)\cdot\sim(q))$, (D2; 3)

(5) $f(1,q)\cdot p\equiv(f(1,1)\cdot p\cdot q\vee f(1,0)\cdot p\cdot\sim(q))$, (4)

(6) $\chi<f,0>(q)\equiv(\chi<f,0>(1)\cdot q\vee\chi<f,0>(0)\cdot\sim(q))$, (a)

(7) $f(0,q)\equiv(f(0,1)\cdot q\vee f(0,0)\cdot\sim(q))$, (6; D2)

(8) $f(0,q)\cdot\sim(p)\equiv(f(0,1)\cdot\sim(p)\cdot q\vee f(0,0)\cdot\sim(p)\cdot\sim(q))$, (7)

(9) $f(1,q)\cdot p\vee f(0,q)\cdot\sim(p)$, (2; 5; 8)

(10) $(\psi<f,q>(1)\cdot p\vee\psi<f,q>(0)\cdot\sim(p))\supset\psi<f,q>(p)$, (1)

(11) $(f(1,q)\cdot p\vee f(0,q)\cdot\sim(p))\supset f(p,q)$, (D1; 10)

$f(p,q)$. (9; 11)

We obtain Theorem 4. Conversely, we shall prove the following

Theorem 5. $[f,p]\{[f,r,s]\{f(1,1)\cdot r\cdot s\vee f(1,0)\cdot r\cdot\sim(s)\vee f(0,1)\cdot\sim(r)\cdot s\vee f(0,0)\cdot\sim(r)\cdot\sim(s))\supset f(r,s)\}\cdot(f(1)\cdot p\vee f(0)\cdot\sim(p))\supset f(p)\}$.

Proof. (1) $[f,r,s]\{(f(1,1)\cdot r\cdot s\vee f(1,0)\cdot r\cdot\sim(s)\vee f(0,1)\cdot\sim(r)\cdot s\vee f(0,0)\cdot\sim(r)\cdot\sim(s))\supset f(r,s)\}$,

(2) $(f(1)\cdot p\vee f(0)\cdot\sim(p))\supset$

(3) $\chi<f,1>(q)\equiv(\chi<f,1>(1)\cdot q\vee\chi<f,1>(0)\cdot\sim(q))$, (a)

(4) $f(1,q)\equiv(f(1,1)\cdot q\vee f(1,0)\cdot\sim(q))$, (D2; 3)

(5) $f(1,q)\cdot p\equiv(f(1,1)\cdot p\cdot q\vee f(1,0)\cdot p\cdot\sim(q))$, (4)

(6) $\chi<f,0>(q)\equiv(\chi<f,0>(1)\cdot q\vee\chi<f,0>(0)\cdot\sim(q))$, (a)

(7) $f(0,q)\cdot\sim(p)\equiv(f(0,1)\cdot\sim(p)\cdot q\vee f(0,0)\cdot\sim(p)\cdot\sim(q))$, (D2; 6)

(8) $[f,p,q]\{(f(1,q)\cdot p\vee f(0,q)\cdot\sim(p))\supset f(p,q)\}$, (1; 5; 7)

(9) $(f(1)\cdot p\vee f(0)\cdot\sim(p))\supset f(p)$, (8)

$f(p)$ (2; 9)

Theorem 5 has been proved.

Thus we have been Theorem 6 from Theorems 4 and 5.

Theorem 6. $[f,p]\{(f(1)\cdot p\vee f(0)\cdot\sim(p))\supset f(p)\}\equiv[f,p,q]\{(f(1,1)\cdot p\cdot q\vee f(1,0)\cdot p\cdot\sim(q)\vee f(0,1)\cdot\sim(p)\cdot q\vee f(0,0)\cdot\sim(p)\cdot\sim(q))\supset f(p,q)\}$.

Therefore we have Theorem 7 from Theorems 3, 6, and the theorem of the propositional calculus:

$$[p,q,r,s]\{(p\equiv q)\cdot(r\equiv s)\supset(p\cdot r\equiv q\cdot s)\},$$

$$[p,p',q,q']\{(p\equiv q)\cdot(p\equiv p')\cdot(q\equiv q')\supset(p'\equiv q')\}.$$

Theorem 7. $[f,p]\{f(p)\equiv(f(1)\cdot p\vee f(0)\cdot\sim(p))\}\equiv[f,p,q]\{f(p,q)\equiv(f(1,1)\cdot p\cdot q\vee f(1,0)\cdot p\cdot\sim(q)\vee f(0,1)\cdot\sim(p)\cdot q\vee f(0,0)\cdot\sim(p)\cdot\sim(q))\}$.

Theorem 7 states that the law of development on functions of one argument is equivalent to that on functions of two arguments. Therefore we complete the proof of the equivalence of the laws of development.

Further we shall prove that the *law* (*b*) *on the limit of a functions* of one argument is equivalent to its corresponding law (b') on a functions of two argument:

(b) $[f]\{[p]\{f(p)\} \equiv f(1)\cdot f(0)\}$,

(b′) $[f]\{[p, q]\{f(p, q)\} \equiv f(1, 1)\cdot f(1, 0)\cdot f(0, 1)\cdot f(0, 0)\}$.

We assert the following

Theorem 8. $[f, q, r]\{[f]\{f(1)\cdot f(0) \supset [p]\{f(p)\}\}\cdot f(q, 1)\cdot f(q, 0) \supset \psi < f, r > (q)\}$.

Proof. (1) $[f]\{f(1)\cdot f(0) \supset [p]\{f(p)\}\}$,

(2) $f(q, 1)$,

(3) $f(q, 0) \supset$

(4) $\chi < f, q > (1)$, (D1; 2)

(5) $\chi < f, q > (0)$, (D1; 3)

(6) $[r]\{\chi < f, q > (r)\}$, (1; 4, 5)

(7) $f(q, r)$, (D1; 6)

$\psi < f, r > (q)$, (D1; 7)

then we complete the proof. Conversely we obtain the following

Theorem 9. $[f, q, r]\{[f]\{f(1)\cdot f(0) \supset [p]\{f(p)\}\}\cdot f(1, 1)\cdot f(1,0) \cdot f(0,1)\cdot f(0, 0) \supset f(q, r)\}$.

Proof. (1) $[f]\{f(1)\cdot f(0) \supset [p]\{f(p)\}\}$,

(2) $f(1, 1)$,

(3) $f(1, 0)$,

(4) $f(0, 1)$,

(5) $f(0, 0) \supset$

(6) $\psi < f, r > (1)$, (Theorem 8; 1; 2; 3)

(7) $\psi < f, r > (0)$, (Theorem 8; 1; 4; 5)

(8) $[q][\psi < f, r > (q)\}$, (1; 6; 7)

$f(q, r)$. (D1; 8)

We have proved Theorem 9. Therefore we obtain Theorem 10 from Theorem 9 and the theorem of the propositional calculus:

$$[p, q, q']\{(p \supset q)\cdot(p \supset q') \supset (p \supset q\cdot q')\}.$$

Theorem 10. $[f]\{[f]\{f(1)\cdot f(0) \supset [p]\{f(p)\}\}\cdot f(1,1)\cdot f(1, 0) \cdot f(0,1)\cdot f(0, 0) \supset [q, r]\{f(q, r)\}\}$.

We shall prove

Theorem 11. $[f]\{[f]\{f(1, 1)\cdot f(1, 0)\cdot f(0, 1)\cdot f(0, 0) \supset [p, q] \{f(p, q)\}\}\cdot f(1)\cdot f(0) \supset f(r)\}\cdot$

Proof. (1) $[f]\{f(1, 1)\cdot f(1, 0)\cdot f(0, 1)\cdot f(0, 0) \supset [p, q]\{f(p, q)\}\}$,

(2) $f(1)$,

(3) $f(0) \supset$

(4) $\omega \Leftarrow f \Rightarrow (1, 1)$, (2)

(5) $\omega \Leftarrow f \Rightarrow (1, 0)$, (2)

(6) $\omega \Leftarrow f \Rightarrow (0, 1)$, (3)

(7) $\omega \Leftarrow f \Rightarrow (0, 0)$, (3)

(8) $[s, t]\{\omega \Leftarrow f \Rightarrow (s, t)\}$, (1; 4; 5: 6; 7)

(9) $\omega \Leftarrow f \Rightarrow (r, r)$, (8)

$f(r)$. (Theorem 10)

We complete the proof. Therefore we obtain Theorem 12 from Theorem 11.

Theorem 12. $[f]\{[f]\{f(1,1) \cdot f(1,0) \cdot f(0,1) \cdot f(0,0) \supset [p,q]\{f(p, q)\}\} \cdot f(1) \cdot f(0) \supset [r]\{f(r)\}\}$.

Accordingly we obtain Theorem 13 from Theorems 10 and 12.

Theorem 13. $[f]\{f(1) \cdot f(0) \supset [p]\{f(p)\}\} \equiv [f]\{f(1, 1) \cdot f(1, 0) \cdot f(0, 1) \cdot f(0, 0) \supset [p, q]\{f(p, q)\}\}$.

By an analogous way, we obtain the Theorems 14 and 15.

Theorem 14. $[f]\{[f]\{[r]\{f(r)\} \supset f(1) \cdot f(0)\} \cdot [p, q]\{f(p, q)\} \subset f(1, 1) \cdot f(1, 0) \cdot f(0, 1) \cdot f(0, 0)\}$.

Theorem 15. $[f]\{[f]\{[p, q]\{f(p, q)\} \supset f(1, 1) \cdot f(1, 0) \cdot f(0, 1) \cdot f(0, 0)\} \cdot [p]\{f(p)\} \supset f(1) \cdot f(0)\}$.

We obtain Theorem 16 from Theorems 14 and 15

Theorem 16. $[f]\{[p]\{f(p)\} \supset f(1) \cdot f(0)\} \equiv [f]\{[p, q]\{f(p, q)\} \supset f(1, 1) \cdot f(1, 0) \cdot f(0, 1) \cdot f(0, 0)\}$.

Then we obtain

Theorem 17. $[f]\{[p]\{f(p)\} \equiv f(1) \cdot f(0)\} \equiv [f]\{[p, q]\{f(p, q)\} \equiv f(1, 1) \cdot f(1, 0) \cdot f(0, 1) \cdot f(0,0)\}$.

Therefore we complete the proof of the equivalence of the law on the limit of a functions in elementary protothetics.

Reference

[1] J. Słupecki: St. Leśniewski's protothetics. Studia Logica, **1**, 44-112 (1953).