160. *A Characterization of Lukasiewiczian Algebra. I*

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(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1967)

In his papers [1], [2], using an algebraic technique, Prof. K. Iséki gave a characterisation of Boolean algebra. In this paper, I shall give a characterization of three-valued Lukasiewicz algebras, which were introduced by Prof. Gr. C. Moisil [3] as models for J. Lukasiewicz three-valued propositional calculus [4].

A L-algebra is a system $\langle X, 0, *, \sim\rangle$ where 0 is an element of a set X, $*$ is a binary operation and $\sim$ is a unary operation on X such that the axioms given below hold. We write $x\leq y$ for $x*y=0$, and $x=y$ for $x\leq y$ and $y\leq x$.

$L1)$ $\quad x*y\leq x,$

$L2)$ $\quad (x*y)*(x*z)\leq z*y,$

$L3)$ $\quad x*(x*(z*(z*y)))\leq z*(z*(y*(y*x))),$

$L4)$ $\quad (x*z)*((x*z)*(y*z))\leq (y*z)*(y*x),$

$L5)$ $\quad x\leq x*(\sim x*x),$

$L6)$ $\quad x*(x*\sim x)\leq \sim(y*(y*\sim y))),$

$L7)$ $\quad x*\sim y\leq y*\sim x,$

$L8)$ $\quad \sim x*y\leq \sim y*x,$

$L9)$ $\quad 0\leq x.$

Further we shall prove some proposition from the axioms $L1$—$L9$.

If we substitute $y*z$ for z in $L2$, then by $L1$, $L9$ we have

$$x*y\leq x*(y*z). \tag{1}$$

In (1) if we put $y=x$, $z=\sim x*x$ and use $L5$, $L9$, then we have

$$x*x=0. \tag{2}$$

By $L1$, $L9$ we have

$$0*x=0. \tag{3}$$

In $L3$ put $z=0$, then by (3), $L2$ we have

$$x=x*0. \tag{4}$$

By $L2$ we have the following lemmas.

Lemma 1. *$x\leq y$ implies $z*y\leq z*x$.*

Lemma 2. *$x\leq y$ and $y\leq z$ imply $x\leq z$.*

Let us put $z=y$ in $L3$, then by $L1$, (2), (4), Lemma 2, we have

$$x*(x*y)\leq y. \tag{5}$$

By $L2$ and Lemma 1 we have

$$u*(z*y)\leq u*((x*y)*(x*z)). \tag{6}$$

In (6) put $x=x*u$, $z=x*z$, $u=((x*u)*y)*(z*u)$ then

$$(((x*u)*y)*(z*u))*((x*z)*y)$$
$$\leq(((x*u)*y)*(z*u))*(((x*u)*y)*((x*u)*(x*z))).$$

The right side is equal to 0 by (6), hence we have the following

(7) $$((x*u)*y)*(z*u)\leq(x*z)*y.$$

In (7) put $y=z$, $z=x*z$, $u=y$, then by (5) we have

(8) $$(x*y)*z\leq(x*z)*y.$$

By (5) and Lemma 1 we have

(9) $$x*y\leq x*(x*(x*y)).$$

If we substitute y for z in $L3$, then by (2), (4) we have

(10) $$x*(x*y)\leq y*(y*(y*(y*x))).$$

By (10), (9), and Lemma 1 we have the following

(11) $$x*(x*y)\leq y*(y*x),$$

hence

(12) $$x*(x*y)=y*(y*x).$$

In $L7$ put $y=\sim x$, then by (2) we have

(13) $$x\leq\sim\sim x.$$

Let us put $x=\sim x$, $y=x$ in $L8$, then, by (2) we have

(14) $$\sim\sim x\leq x,$$

hence

(15) $$x=\sim\sim x.$$

By (15), $L7$ we have

(16) $$x*y=\sim y*\sim x,$$

hence we have the following

Lemma 3. *$x\leq y$ implies $\sim y\leq\sim x$.*

By $L3$, (12) we have

(17) $$x*(x*(y*(y*z)))\leq z*(z*(y*(y*x))).$$

If we put $x=z$, $z=x$ in the formula above, we have

(18) $$z*(z*(y*(y*(y*x)))\leq x*(x*(y*(y*z))),$$

hence

(19) $$x*(x*(z*(z*y)))=z*(z*(y*(y*x))).$$

By $L1$, $L5$ we have

(20) $$x=x*(\sim x*x).$$

By (8) we have the following

Lemma 4. *$x*y\leq z$ implies $x*z\leq y$.*

By $L2$ and Lemma 4 we have

(21) $$(x*y)*(z*y)\leq x*z$$

hence

Lemma 5. *$x\leq z$ implies $x*y\leq z*y$.*

Let $z\leq x$. By Lemma 1 we have $\sim x*x\leq\sim x*z$ and by Lemma 3, $\sim x\leq\sim z$ and applying Lemma 5, we have $\sim x*z\leq\sim z*z$.

Hence we have the following

Lemma 6. *$z\leq x$ implies $\sim x*x\leq\sim z*z$.*

By $L4$ we have

Lemma 7. *$y*z\leq y*x$ and $x*z\leq y*z$ imply $x\leq z$.*

If we define $x\cap y=y*(y*x)$, $x\cup y=\sim(\sim y*(x*y))$, $\mu x=\sim(\sim x*x)$ and $1=\sim 0$, then we shall prove that a L-algebra is a three-valued Lukasiewicz algebra.

I. $x\cap y=y\cap x$; $x\cup y=y\cup x$

are proved by $x\cap y=y*(y*x)=x*(x*y)=y\cap x$ and $x\cup y=\sim(\sim y*(x*y))=\sim(\sim y*(\sim y*\sim x))=\sim(\sim x*(\sim x*\sim y))=\sim(\sim x*(y*x))=y\cup x$.

II. $x\cap(x\cup y)=x$; $x\cup(x\cap y)=x$

are proved by $x\cap(x\cup y)=(y\cup x)\cap x=x*(x*\sim(\sim x*(y*x)))=x*((\sim x*(y*x))*\sim x)=x*0=x$ and $x\cup(x\cap y)=(y\cap x)\cup x=\sim(\sim x*((x*(x*y))*x))=\sim(\sim x*0)=\sim\sim x=x$.

III. $x\cap(y\cap z)=(x\cap y)\cap z$; $x\cup(y\cup z)=(x\cup y)\cup z$

follow from $x\cap(y\cap z)=(y\cap z)\cap x=x*(x*(z*(z*y)))=z*(z*(y*(y*x)))=(x\cap y)\cap z$ and $x\cup(y\cup z)=(y\cup z)\cup x=\sim(\sim x*(\sim(\sim z*(y*z))*x)=\sim(\sim x*(\sim x*(\sim z*(\sim z*\sim y))))=\sim(\sim x*(\sim x*(\sim y*(\sim y*\sim z))))=\sim(\sim z*(\sim z*(\sim y*(\sim y*\sim x))))=\sim(\sim z*(\sim(\sim y*(x*y))*z))=(x\cup y)\cup z$.

IV. $x\leq y\rightleftarrows x\cap y=x$.

If $x\leq y$, i.e. $x*y=0$, then $x\cap y=y*(y*x)=x*(x*y)=x*0=x$. Conversely, if $x\cap y=x$ i.e. $y*(y*x)=x$, then $x*(x*(x*y))=0$, hence $x*y=0$.

By I—III a L-algebra is a lattice, and by IV, $\leq$ is the order.

V. $x\cap y\leq z$ and $x\leq y\cup z$ imply $x\leq z$.

If $x\leq y\cup z$, then $x\leq\sim(\sim z*(y*z))$ i.e. $x*z\leq y*z$ and if $x\cap y\leq z$, then $y*(y*x)\leq z$ i.e. $y*z\leq y*x$.

Applying the Lemma 7 we have V.

By I—V we have (see [5] p. 137) that a L-algebra is a distributive lattice.

VI. $x\cup 1=1$

is proved by $x\cup 1=\sim(\sim 1*(x*1))=\sim(0*(x*1))=\sim 0=1$.

VII. $\sim\sim x=x$

is proposition (15).

VIII. $\sim(x\cap y)=\sim x\cup\sim y$

follows from $\sim(x\cap y)=\sim(y*(y*x))=\sim(y*(\sim x*\sim y))=\sim x\cup\sim y$.

IX. $x\cap\sim x\leq y\cup\sim y$

follows from axiom $L6$.

According to I—IX a L-algebra is a Kleene algebra.

X. $\sim x\cup\mu x=1$

is proved by $\sim x\cup\mu x=\mu x\cup\sim x=\sim(\sim x*x)\cup\sim x=\sim((\sim x*x)\cap x)=\sim(x*(x*(\sim x*x)))=\sim 0=1$.

XI. $x\cap\sim x=\sim x\cap\mu x$

follows from $x \cap \sim x = \sim x*(\sim x*x) = \sim(\sim x*x)*x = \sim(\sim x*x) *(x*(\sim x*x)) = \sim(\sim x*x)*(\sim(\sim x*x)*\sim x) = \sim x \cap \mu x$.

XII. $\mu(x \cap y) \leq \mu x \cap \mu y$.

If we substitute $y*(y*x)$ for z in Lemma 6, then we have $\sim x*x \leq \sim(y*(y*x))*(y*(y*x))$ hence $\sim(\sim(y*(y*x))*(y*(y*x))) \leq \sim(\sim x*x)$ whence $\mu(x \cap y) \leq \mu x$. For $z = y*(y*x)$ and $x = y$ in Lemma 6 analogously we have $\mu(x \cap y) \leq \mu y$, hence XII.

A Kleene algebra such that the conditions X—XII are satisfied is a three-valued Lukasiewicz algebra [6].

Hence we have the following

Theorem. A L-algebra is a three-valued Lukasiewicz algebra.

References

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