159. *On Many-Valued Lukasiewicz Algebras*

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(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1967)

The many-valued Lukasiewicz algebras were introduced by Prof. Gr. C. Moisil [2] as models for J. Lukasiewicz' many-valued propositional calculus.

In this note, we give a system with a smaller number of axioms for the many-valued Lukasiewicz algebras. Also is considered a generalization for the notion of strict chrysippien element and we give conditions in which a many-valued Lukasiewicz algebra is a Kleene algebra.

1. The many-valued Lukasiewicz algebra is a lattice L which satisfies [2] the following axioms:

$L1$) L is a distributive lattice with first and last element.

$L2$) L has an involutive duality named negation:

$$N(x\cap y)=Nx\cup Ny,$$
$$N(x\cup y)=Nx\cap Ny,$$
$$NNx=x.$$

$L3$) L has $n-1$ endomorphisms $\sigma_1, \cdots, \sigma_{n-1}$:

$$\sigma_i(x\cup y)=\sigma_i x\cup\sigma_i y,$$
$$\sigma_i(x\cap y)=\sigma_i x\cap\sigma_i y.$$

$L4$) The elements $\sigma_i x$ are chrysippien

$$\sigma_i x\cup N\sigma_i x=1,$$
$$\sigma_i x\cap N\sigma_i x=0.$$

$L5$) The elements $\sigma_i x$ form a linear ordered lattice

$$\sigma_1 x\subset\sigma_2 x\subset\cdots\subset\sigma_{n-1}x.$$

$L6$) There is the relation

$$\sigma_i\sigma_j x=\sigma_j x \quad \text{for any } i, j.$$

$L7$) There is the relation

$$\sigma_i Nx=N\sigma_j x, \quad \text{where } j=\varphi(i).$$

$L8$) If $\sigma_i x=\sigma_i y$ $(i=1,\cdots,n-1)$ then $x=y$.

$L8$ is called the determination principle.

Prof. Gr. C. Moisil proves [3] that

(1) $$\sigma_i 0=0, \quad \sigma_i 1=1 \quad (1\leq i\leq n-1).$$

If we introduce [2] the Lagrange's fonctions

(2) $$\lambda_i x=\sigma_{n-i}x\cap N\sigma_{n-i-1}x \qquad (0\leq i\leq n-1),$$

where $\sigma_0 x=0$, $\sigma_n x=1$, then [3] the following formulas hold:

(3) $$\sigma_{n-i}x=\lambda_i x\cup\lambda_{i+1}x\cup\cdots\cup\lambda_{n-1}x,$$

(4) $$N\sigma_{n-i}x=\lambda_0 x\cup\lambda_1 x\cup\cdots\cup\lambda_{i-1}x,$$

(5) $$\lambda_i x\cap\lambda_j x=0\qquad(i\neq j),$$

(6) $$\lambda_0 x\cup\lambda_1 x\cup\cdots\cup\lambda_{n-1}x=1.$$

2. **Theorem 1.** *A many-valued Lukasiewicz algebra is a system* $\langle L, 1, N, \sigma_1, \cdots, \sigma_{n-1}, \cap\rangle$, *where* 1 *is not postulated as last element, verifying the following axioms*:

$L^*1)$ $\quad x\cap N(Nx\cap Ny)=x,$

$L^*2)$ $\quad x\cap N(Ny\cap Nz)=N(N(z\cap x)\cap N(y\cap x)),$

$L^*3)$ $\quad N(\sigma_i Nx\cap\sigma_j x)=1,\quad j=\varphi(i),$

$L^*4)$ $\quad \sigma_i\sigma_j(x\cap y)=\sigma_j x\cap\sigma_j y,$ *for any* i, j,

$L^*5)$ $\quad \sigma_i Nx\cap\sigma_k Nx=N\sigma_j x,\quad i\leq k, j=\varphi(i),$

$L^*6)$ $\quad$ *if* $\sigma_i x=\sigma_i y\ (1\leq i\leq n-1)$, *then* $x=y$.

It is easily to prove the necessity. We prove now the converse implication.

R. Marrona has proved [4] that the Morgan algebras can be defined with the axioms L^*1 and L^*2, where, by definition

(7) $$x\cup y=N(Nx\cap Ny).$$

Let $x=y$ in L^*4, then we have

(8) $$\sigma_i\sigma_j x=\sigma_j x\quad\text{for any } i, j.$$

From L^*4 and (8) we have

(9) $$\sigma_i(x\cap y)=\sigma_i x\cap\sigma_i y.$$

For $i=k$, by L^*5 we can deduce

(10) $$\sigma_i Nx=N\sigma_j x,\quad j=\varphi(i).$$

By (10) and L^*5 we have

(11) $$\sigma_1 x\subset\sigma_2 x\subset\cdots\subset\sigma_{n-1}x.$$

By L^*3 and (10) it follows that

(12) $$\sigma_i x\cup N\sigma_i x=1$$

and applying the negation

(13) $$\sigma_i x\cap N\sigma_i x=0$$

where we put $0=N1$.

From (9) and (10) we have

$$\sigma_i(x\cup y)=\sigma_i N(Nx\cap Ny)=N\sigma_j(Nx\cap Ny)$$
$$=N\sigma_j Nx\cup N\sigma_j Ny=\sigma_i x\cup\sigma_i y,$$

hence

(14) $$\sigma_i(x\cup y)=\sigma_i x\cup\sigma_i y.$$

If $\sigma_i x\subset\sigma_i y\ (1\leq i\leq n-1)$, then $\sigma_i(x\cap y)=\sigma_i x$, hence from L^*6 we have $x\cap y=x$, i.e.

(15) $$\sigma_i x\subset\sigma_i y\quad(1\leq i\leq n-1)\quad\text{imply } x\subset y.$$

By (8), (11) we have $\sigma_i\sigma_1 x\subset\sigma_i x\subset\sigma_i\sigma_{n-1}x$ and according to (15) we have

(16) $$\sigma_1 x\subset x\subset\sigma_{n-1}x.$$

The last relation and (12), (13) give

$$x \cup 1 = x \cup \sigma_{n-1}x \cup N\sigma_{n-1}x = \sigma_{n-1}x \cup N\sigma_{n-1}x = 1,$$
$$x \cap 0 = x \cap \sigma_1 x \cap N\sigma_1 x = \sigma_1 x \cap N\sigma_1 x = 0,$$

hence 0 and 1 are respectively the first and the last element.

Therefore the proof of our statement is complete.

Lemma 1. *In a many-valued Lukasiewicz algebras the following relations hold.*

(17) $$\sigma_{n-1}(x \cup Nx) = 1,$$

(18) $$\sigma_1(x \cap Nx) = 0.$$

From

$$\sigma_j 1 = 1 = \sigma_i x \cup N\sigma_i x = \sigma_i x \cup \sigma_j Nx = \sigma_j(\sigma_i x \cup Nx) \subset \sigma_j(\sigma_{n-1}x \cup Nx),$$

applying the Moisil determination principle we have $\sigma_{n-1}x \cup Nx = 1$. But $\sigma_{n-1}(x \cup Nx) \supset \sigma_{n-1}x \cup Nx$, hence (17). From

$$\sigma_j 0 = 0 = \sigma_i x \cap N\sigma_i x = \sigma_i x \cap \sigma_j Nx = \sigma_j(\sigma_i x \cap Nx) \supset \sigma_j(\sigma_1 x \cap Nx),$$

applying the Moisil determination principle we have $\sigma_1 x \cap Nx = 0$. But $\sigma_i(x \cap Nx) \subset \sigma_1 x \cap Nx$, hence (18).

Theorem 2. *A many-valued Lukasiewicz algebra with $\varphi(i) \equiv n - i$ (mod n) is a Kleene algebra.*

If n is odd, we have the followings:

$$\sigma_{n-2}(x \cup Nx) = \sigma_{n-2}x \cup \sigma_{n-2}Nx = \sigma_{n-2}x \cup N\sigma_2 x$$
$$= \lambda_2 x \cup \lambda_3 x \cup \cdots \cup \lambda_{n-1}x \cup \lambda_0 x \cup \lambda_1 x \cup \cdots \cup \lambda_{n-3}x = 1,$$
$$\sigma_{n-3}(x \cup Nx) = \sigma_{n-3}x \cup \sigma_{n-3}Nx = \sigma_{n-3}x \cup N\sigma_3 x$$
$$= \lambda_3 x \cup \lambda_4 x \cup \cdots \cup \lambda_{n-1}x \cup \lambda_0 x \cup \lambda_1 x \cup \cdots \cup \lambda_{n-4}x,$$

. . . .

$$\sigma_{\frac{n+1}{2}}(x \cup Nx) = \sigma_{\frac{n+1}{2}}x \cup \sigma_{\frac{n+1}{2}}Nx = \sigma_{\frac{n+1}{2}}x \cup N\sigma_{\frac{n-1}{2}}x$$
$$= \lambda_{\frac{n-1}{2}}x \cup \cdots \cup \lambda_{n-1}x \cup \lambda_0 x \cup \cdots \cup \lambda_{\frac{n-1}{2}}x = 1,$$
$$\sigma_{\frac{n-1}{2}}(x \cap Nx) = \sigma_{\frac{n-1}{2}}x \cap \sigma_{\frac{n-1}{2}}Nx = \sigma_{\frac{n-1}{2}}x \cap N\sigma_{\frac{n+1}{2}}x$$
$$= (\lambda_{\frac{n+1}{2}}x \cup \cdots \cup \lambda_{n-1}x) \cap (\lambda_0 x \cup \cdots \cup \lambda_{\frac{n-3}{2}}x) = 0,$$

. . . .

$$\sigma_3(x \cap Nx) = \sigma_3 x \cap \sigma_3 Nx = \sigma_3 x \cap N\sigma_{n-3}x$$
$$= (\lambda_{n-3}x \cup \lambda_{n-2}x \cup \lambda_{n-1}x) \cap (\lambda_0 x \cup \lambda_1 x \cup \lambda_2 x) = 0,$$
$$\sigma_2(x \cap Nx) = \sigma_2 x \cap \sigma_2 Nx = \sigma_2 x \cap N\sigma_{n-2}x$$
$$= (\lambda_{n-2}x \cup \lambda_{n-1}x) \cap (\lambda_0 x \cup \lambda_1 x) = 0.$$

From the preceding relations and Lemma 1 we obtain $\sigma_i(x \cap Nx) \subset \sigma_i(y \cup Ny)$ $(1 \leq i \leq n-1)$ and applying the Moisil determination principle, it follows that $x \cap Nx \subset y \cup Ny$.

If n is even, we can analogously prove the above formulas.

Hence we complete the proof of Theorem 2.

For the three-valued Lukasiewicz algebra, the result was obtained by A. Monteiro [5].

Lemma 2. *In a many-valued Lukasiewicz algebra, $x \cap y = 0$ if and only if $\sigma_i y \subset N\sigma_i x$ $(1 \leq i \leq n-1)$.*

If $x \cap y = 0$, then $\sigma_i x \cap \sigma_i y = 0$ $(1 \leq i \leq n-1)$, hence $(\sigma_i x \cap \sigma_i y) \cup N\sigma_i x = N\sigma_i x$ whence $\sigma_i y \subset N\sigma_i x$.

Conversely, if $\sigma_i y \subset N\sigma_i x$ $(1 \leq i \leq n-1)$, then $\sigma_i x \cap \sigma_i y \subset \sigma_i y \cap N\sigma_i x$, hence $\sigma_i(x \cap y) = 0 = \sigma_i 0$ and applying the Moisil determination principle, we have $x \cap y = 0$.

Definition. An element m of the many-valued Lukasiewicz algebra is called strict chrysippien [1], if for any x, $m \cup x$ is chrysippien.

Theorem 3. *m is strict chrysippien if and only if $\sigma_i m \supset \sigma_i(x \cap Nx)$ where $j = \varphi(i)$ and x is arbitrary.*

Let m be strict chrysippien, then, for every x, $(m \cup x) \cap N(m \cup x) = 0$. If we put $x = 0$ in the preceding relation, we have $m \cap Nm = 0$ whence $x \cap Nx \cap Nm = 0$ for every x. By Lemma 2, we have $N\sigma_i Nm \supset \sigma_i(x \cap Nx)$ for any i, hence $\sigma_j m \supset \sigma_i(x \cap Nx)$, $j = \varphi(i)$.

Conversely, if $\sigma_j m \supset \sigma_i(x \cap Nx)$, where $j = \varphi(i)$ and x is arbitrary then $N\sigma_i Nm \supset \sigma_i(x \cap Nx)$ and by Lemma 2 we have $x \cap Nx \cap Nm = 0$. If we put $x = m$ in the preceding relation we have $m \cap Nm = 0$ hence $(m \cup x) \cap N(m \cup x) = 0$ for every x.

References

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