157. On Normal Analytic Sets. II

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(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1967)

I have studied conditions for an analytic set being normal and obtained the following [1].[1]

Theorem 1. *If $\sum$ is normal at 0, then $\sum$ satisfies the conditions (α) and (β). Moreover, when $\sum$ is principal, $\sum$ is normal at 0 if and only if $\sum$ satisfies the conditions (α) and (β).*

The two conditions in Theorem 1 are the following.

Condition (α).[2] *Let (x^0, y^0) be a point sufficiently near 0, such that $\delta(x^0)\neq 0$, $f(x^0, y^0)=0$. Let*

$$z_\mu = z_\mu^{(i)}(x, y)=\sum_{\nu=0}^{\infty} c_\nu^{(i,\mu)}(x)(y-\varphi(x))^{\frac{\nu}{pi}},\ 1\leqslant\mu\leqslant e,\ 1\leqslant i\leqslant\kappa,$$

be the systems of Puiseux-series, attached to (x^0, y^0). Then, for i, j, $i\neq j$, there exists an index μ, $1\leqslant\mu\leqslant e$, such that we have $c_0^{(i,\mu)}(x^0)\neq c_0^{(j,\mu)}(x^0)$.

Condition (β). *Let (x^0, y^0) be a point sufficiently near 0, such that $\delta(x^0)\neq 0$, $f(x^0, y^0)=0$. Let*

$$z_\mu = z_\mu(x, y)=\sum_{\nu=0}^{\infty} c_\nu^{(\mu)}(x)(y-\varphi(x))^{\frac{\nu}{p}},\ 1\leqslant\mu\leqslant e,$$

be a system of Puiseux-series, attached to (x^0, y^0), such that $p>1$. Then we have $c_1^{(\mu)}(x)\not\equiv 0$ for an index μ, $1\leqslant\mu\leqslant e$.

The notations given in [1] are used in the above statements and will be in the following.

In this note, two conditions are newly introduced to improve Theorem 1. Consider the following.

Condition (γ). *Let (x^0, y^0) be a point sufficiently near 0, such that $\delta(x^0)\neq 0$, $f(x^0, y^0)=0$. Let*

$$z_\mu = z_\mu^{(i)}(x, y)=\sum_{\nu=0}^{\infty} c_\nu^{(i,\mu)}(x)(y-\varphi(x))^{\frac{\nu}{pi}},\ 1\leqslant\mu\leqslant e,\ 1\leqslant i\leqslant\kappa,$$

be the systems of Puiseux-series, attached to (x^0, y^0). Then, for i, j, $i\neq j$, there exists an index μ, $1\leqslant\mu\leqslant e$, such that we have $c_0^{(i,\mu)}(x)\not\equiv c_0^{(j,\mu)}(x)$.

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1) Prof. K. Kasahara has kindly pointed out, with a counter example, the incredibility of Theorem 2, [1]. And I found out several errors in [1]. In [1], the propositions and theorems need the assumption that $\sum$ is *principal*, except for Propositions 3, 4: the reader would take care of the fact that, even if $\sum$ is nonprincipal, the "only if" parts of Propositions 1, 2 however are true. Theorem 1, [1] should therefore be corrected as in the present paper.

2) The condition (α) in [1] was incorrect and should be thus revised.

Condition (δ). *Let (x^0, y^0) be a point sufficiently near 0, such that $\delta(x^0)\neq 0$, $f(x^0, y^0)=0$. Let*

$$z_\mu = z_\mu(x, y) = \sum_{\nu=0}^{\infty} c_\nu^{(\mu)}(x)(y-\varphi(x))^{\frac{\nu}{p}},\ 1\leqslant \mu \leqslant e,$$

be a system of Puiseux-series, attached to (x^0, y^0), such that $p>1$. Then we have $c_1^{(\mu)}(x^0)\neq 0$ for an index μ. $1\leqslant \mu\leqslant e$.

We see that 1) (α) induces (γ) and 2) (δ) induces (β). And we have first

Proposition 6. *If $\sum$ is normal at 0, it satisfies the condition (δ).*

Proof. Suppose that $\sum$ is normal at 0: then, by Prop. 3, [1], we see that $\sum$ satisfies the condition (α). Suppose that $\sum$ does not satisfy the condition (δ); then there exist a point (x^0, y^0, z^0), close to 0 and satisfying $\delta(x^0)\neq 0$, $f(x^0, y^0)=0$, and a system passing through (x^0, y^0, z^0):

$$z_\mu = z_\mu(x, y) = \sum_{\nu=0}^{\infty} c_\nu^{(\mu)}(x)(y-\varphi(x))^{\frac{\nu}{p}},\ 1\leqslant \mu \leqslant \mathrm{e},$$

with $p>1$ and $c_1^{(\mu)}(x^0)=0$, $1\leqslant\mu\leqslant \mathrm{e}$. The system describes $\sum$ completely in a neighborhood of (x^0, y^0, z^0).

Consider the function $h=(y-\varphi(x))^{\frac{1}{p}}$ which is holomorphic on $\sum$ in a neighborhood of (x^0, y^0, z^0). By hypothesis, there exists a function $H(x, y, z)$ holomorphic in the space (x, y, z) and representing h in a small closed polydisc U about (x^0, y^0, z^0). Expand H into power-series:

$$H(x, y, z) = \sum_{i=0}^{\infty} b_i(x, z)(y-\varphi(x))^i,$$

where $b_i(x, z)$ are holomorphic in the polydisc $U_1=\{(x, z)\mid (x, y, z)\in U\}$. We have then

$$(1)\qquad (y-\varphi(x))^{\frac{1}{p}} = b_0(x, z) + O((y-\varphi(x)))\qquad \text{for}\quad (x, y, z)\in \textstyle\sum\cap U.$$

Let expand $b_0(x, z)$ into power-series:

$$b_0(x, z) = \sum_{i_1,\cdots,i_e=0}^{\infty} b_{i_1\cdots i_e}(x)(z_1-c_0^{(1)}(x))^{i_1}\cdots(z_e-c_0^{(e)}(x))^{i_e},$$

where $b_{i_1}\cdots_{i_e}(x)$ are holomorphic in the polydisc $U_2=\{x\mid (x, z)\in U_1\}$. Here we have $b_{0\cdots 0}(x)\equiv 0$, since we have

$$b_0(x, z)=0 \quad \text{for}\quad z_\mu = c_0^{(\mu)}(x),\ 1\leqslant\mu\leqslant e,\ (x, z)\in U_1.$$

Consequently, for $(x, y, z)\in\sum\cap U$, we have

$$(2)\qquad b_0(x, z) = C(x)(y-\varphi(x))^{\frac{1}{p}} + O((y-\varphi(x))^{\frac{2}{p}}).$$

We see that $C(x^0)=0$, since $c_1^{(\mu)}(x^0)=0$, $1\leqslant\mu\leqslant e$. From (1) and (2), we have

$$(y-\varphi(x))^{\frac{1}{p}} = C(x)(y-\varphi(x))^{\frac{1}{p}} + O((y-\varphi(x))^{\frac{2}{p}})$$

for (x, y) near (x^0, y^0); this is impossible at $x=x^0$. Q.E.D.

From Prop. 3, [1] and Prop. 6, we have

Theorem 3. *If $\sum$ is normal at* 0, *it satisfies the conditions* (α) *and* (δ).

Theorem 1 induces

Corollary. *Suppose that $\sum$ is principal. Then $\sum$ is normal at* 0 *if and only if $\sum$ satisfies the conditions* (α) *and* (δ).

Remark. If $\sum$ is principal, then 1) (α) is equivalent to (γ) and 2) (β) is equivalent to (δ). Both of 1) and 2) are proved mainly by Lemme 1, p. 139, [2].

In general cases, we have

Theorem 4. *Let $\sum_0$ be the set of non-normal points of $\sum$. Then the dimension of $\sum_0$ at* 0 *does not exceed $d-2$, if and only if $\sum$ satisfies the conditions* (β) *and* (γ).

Proof. 1) Suppose that $\sum$ satisfies these conditions and $\sum_0$ has dimension $>d-2$ at 0. Then, near 0, there exists a regular point (x^0, y^0, z^0) of $\sum_0$, at which $\sum_0$ has dimension $>d-2$. Let $P_\mu(x, y, t)$, $1<\mu\leqslant e$, be distinguished pseudo-polynomials in t, such that $\sum\cap C$ is contained in the set

$$P(x, y, z_1)=P_2(x, y, z_2)=\cdots=P_e(x, y, z_e)=0.$$

Then, if, in a neighborhood U of (x^0, y^0, z^0), $\sum_0$ is contained in the set $\{\delta(x)=0\}$, we have necessarily

$$\textstyle\sum_0\cap U\subset\{\delta(x)=f(x, y)=P(x, y, z_1)=\cdots=P_e(x, y, z_e)=0\},$$

the second member of which has dimension $\leqslant d-2$: this is impossible. Consequently there exists a regular point (x^1, y^1, z^1) of $\sum_0$, close to (x^0, y^0, z^0) and such that $\delta(x^1)\neq 0$, $f(x^1, y^1)=0$; $\sum_0$ has dimension $>d-2$ at (x^1, y^1, z^1). Let

$$z_\mu=z_\mu^{(i)}(x, y)=\sum_{\nu=0}^{\infty}c_\nu^{(i,\mu)}(x)(y-\varphi(x))^{\frac{\nu}{p_i}},\ 1\leqslant\mu\leqslant e,\ 1\leqslant i\leqslant\kappa,$$

be the systems attached to (x^1, y^1) and let

$$\delta'(x)=\delta(x)\prod(c_0^{(i,\mu)}(x)-c_0^{(j,\mu)}(x)),$$

where the product $\prod$ is taken over those i, j, μ such that $i\neq j$ and $c_0^{(i,\mu)}(x)\not\equiv c_0^{(j,\mu)}(x)$: if $\kappa=1$, we set $\delta'(x)=\delta(x)$.

We can prove that, in any neighborhood of (x^1, y^1, z^1), $\sum_0$ is not contained in the set $\{\delta'(x)=0\}$, as in the proof of $\sum_0\cap U\not\subset\{\delta(x)=0\}$. Hence there exists a point $(x^2, y^2, z^2)\in\sum_0$, close to (x^1, y^1, z^1) and such that $\delta'(x^2)\neq 0$, $f(x^2, y^2)=0$; $\sum_0$ is regular and has dimension $>d-2$ at (x^2, y^2, z^2). The systems attached to (x^2, y^2) are given by those which are attached to (x^1, y^1): $\delta'(x^2)\neq 0$ induces that only one of those systems passes through (x^2, y^2, z^2). Let

$$z_\mu=z_\mu(x, y)=\sum_{\nu=0}^{\infty}c_\nu^{(\mu)}(x)(y-\varphi(x))^{\frac{\nu}{p}},\ 1\leqslant\mu\leqslant e,$$

be such a system. We have $p>1$, since $(x^2, y^2, z^2)\in\sum_0$ Accordingly, we have $c_1^{(\mu_0)}(x)\not\equiv 0$ for an index μ_0, and, in a neighborhood of (x^2, y^2, z^2), $\sum_0$ is contained in the set

$$c_1^{(\mu_0)}(x)=0,\ y=\varphi(x),\ z_\mu=c_0^{(\mu)}(x),\ 1\leqslant\mu\leqslant e,$$

which is empty or of $d-2$ dimension at (x^2, y^2, z^2). This is a contradiction.

2) Suppose that $\sum$ does not satisfy the condition (γ); then there exist a point $(x^0, y^0, z^0)\in\sum$, close to 0 and such that $\delta(x^0)\neq 0$, $f(x^0, y^0)=0$, and two systems attached to (x^0, y^0, z^0):

$$z_\mu=z_\mu^{(i)}(x, y)=\sum_{\nu=0}^{\infty}c_\nu^{(i,\mu)}(x)(y-\varphi(x))^{\frac{\nu}{p_i}},\ 1\leqslant\mu\leqslant e,\ i=1, 2,$$

with $c_0^{(1,\mu)}(x)\equiv c_0^{(2,\mu)}(x)$, $1\leqslant\mu\leqslant e$. At each point of the set

$$\textstyle\sum_0':\ y=\varphi(x),\ z_\mu=z_\mu^{(1)}(x, y),\ 1\leqslant\mu\leqslant e,$$

$\sum$ has at least two irreducible components. Consequently $\sum_0$ has dimension $>d-2$ at (x^0, y^0, z^0) and therefore at 0.

If $\sum$ does not satisfy the condition (β), then there exist a point $(x^0, y^0, z^0)\in\sum$, close to 0 and such that $\delta(x^0)\neq 0$, $f(x^0, y^0)=0$ and a system attached to (x^0, y^0, z^0):

$$z_\mu=z_\mu(x, y)=\sum_{\nu=0}^{\infty}c_\nu^{(\mu)}(x)(y-\varphi(x))^{\frac{\nu}{p}},\ 1\leqslant\mu\leqslant e,$$

with $p>1$ and $c_1^{(\mu)}(x)\equiv 0$, $1\leqslant\mu\leqslant e$. At each point of the set

$$\textstyle\sum_0'':\ y=\varphi(x),\ z_\mu=z_\mu(x, y),\ 1\leqslant\mu\leqslant e,$$

$\sum$ is not normal, as we have seen in the proof of Prop. 6. This implies that $\sum_0$ has dimension $>d-2$ at (x^0, y^0, z^0) and therefore at 0.

References

[1] I. Kimura: On normal analytic sets. Proc. Japan Acad., **43** (6), 464-468 (1967).
[2] K. Oka: Sur les fonctions analytiques de plusieurs variables. Iwanami Shoten (1961).