151. *A Generalization of Curry's Theorem*

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1. Introduction. It is well-known that [3] Glivenko obtained a reduction of the classical proposition logic ***LKS*** to the intuitionistic proposition logic ***LJS*** by putting *double negation* in front of each proposition. Thereafter, [1] Curry, as generalization of the Glivenko theorem above, proved:

$$\vdash_{LKS}\mathfrak{A} \text{ if and only if } \vdash_{LJS}\neg\mathfrak{A}\rightarrow\mathfrak{A};$$
$$\vdash_{LDS}\mathfrak{A} \text{ if and only if } \vdash_{LMS}\neg\mathfrak{A}\rightarrow\mathfrak{A},$$

where ***LM*** is the minimal logic introduced by [5] Johansson which has one axiom $(\mathfrak{A}\rightarrow\mathfrak{B})\rightarrow((\mathfrak{A}\rightarrow\neg\mathfrak{B})\rightarrow\neg\mathfrak{A})$ for negation, and ***LD*** is the logic obtained from ***LM*** by assuming further $\mathfrak{A}\vee\neg\mathfrak{A}$, or $(\neg\mathfrak{A}\rightarrow\mathfrak{A})\rightarrow\mathfrak{A}$ (see [2] Curry).

[6] Kleene[1] and [7] Kuroda generalized the Glivenko theorem to predicate logics, namely to a reduction of the classical predicate logic ***LK*** to the intuitionistic predicate logic ***LJ***, essentially by means of *double negation*.

However, the reductions given by them, may be called reductions of ***LK*** to ***LM***. Namely, we can obtain reductions of ***LK*** to ***LM*** by their transformations. On the other hand, the Glivenko theorem does not hold true between ***LKS*** and ***LMS***. Accordingly, it seems natural to ask whether there is a transformation which reduces ***LK*** to ***LJ***, not to ***LM***, and which reduces ***LD*** to ***LM***, as has been done for proposition logics by Curry.

In the following, the authors define a transformation "$_{[\curlywedge]}$", a modification of Curry's transformation ($\mathfrak{A}$ into $\neg\mathfrak{A}\rightarrow\mathfrak{A}$), by means of which we can solve these problems in the affirmative. The authors would like to express their thanks to Prof. K. Ono for his kind guidance and encouragement.[2]

2. Definition of the transformation. The transformation "$_{[\curlywedge]}$" is defined recursively as follows:

(1) If $\mathfrak{P}$ is an elementary formula, $\mathfrak{P}_{[\curlywedge]}\equiv(\mathfrak{P}\rightarrow\curlywedge)\rightarrow\mathfrak{P}$.

(2) If $\mathfrak{A}$ and $\mathfrak{B}$ are formulas,

$$(\mathfrak{A}\rightarrow\mathfrak{B})_{[\curlywedge]}\equiv((\mathfrak{A}_{[\curlywedge]}\rightarrow\mathfrak{B}_{[\curlywedge]})\rightarrow\curlywedge)\rightarrow(\mathfrak{A}_{[\curlywedge]}\rightarrow\mathfrak{B}_{[\curlywedge]}),$$

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1) cf. [4] Gödel. In this paper reductions are given for proposition logic and number theory formulated by Herbrand.

2) Our investigation was originally intended to obtain an interpretation of ***LD*** in ***LO*** under significant suggestion of Prof. K. Ono. See [8] Ono.

$$(\mathfrak{A}\wedge\mathfrak{B})_{[\text{人}]}\overset{=}{<}((\mathfrak{A}_{[\text{人}]}\wedge\mathfrak{B}_{[\text{人}]})\rightarrow\text{人})\rightarrow(\mathfrak{A}_{[\text{人}]}\wedge\mathfrak{B}_{[\text{人}]}),$$
$$(\mathfrak{A}\vee\mathfrak{B})_{[\text{人}]}\overset{=}{<}((\mathfrak{A}_{[\text{人}]}\vee\mathfrak{B}_{[\text{人}]})\rightarrow\text{人})\rightarrow(\mathfrak{A}_{[\text{人}]}\vee\mathfrak{B}_{[\text{人}]}).$$

(3) If x is a variable and $\mathfrak{A}(x)$ is a formula,

$$((x)\mathfrak{A}(x))_{[\text{人}]}\overset{=}{<}((x)\mathfrak{A}_{[\text{人}]}(x)\rightarrow\text{人})\rightarrow(x)\mathfrak{A}_{[\text{人}]}(x),$$
$$((\exists x)\mathfrak{A}(x))_{[\text{人}]}\overset{=}{<}((\exists x)\mathfrak{A}_{[\text{人}]}(x)\rightarrow\text{人})\rightarrow(\exists x)\mathfrak{A}_{[\text{人}]}(x).$$

In ***LM***, negation can be defined by the constant proposition 人, i.e., $\neg\mathfrak{A}\equiv(\mathfrak{A}\rightarrow\text{人})$. Therefore, in the definition above, $\mathfrak{P}_{[\text{人}]}$, $(\mathfrak{A}\rightarrow\mathfrak{B})_{[\text{人}]}$, $(\mathfrak{A}\wedge\mathfrak{B})_{[\text{人}]}$, $(\mathfrak{A}\vee\mathfrak{B})_{[\text{人}]}$, $((x)\mathfrak{A}(x))_{[\text{人}]}$, and $((\exists x)\mathfrak{A}(x))_{[\text{人}]}$ are identified to $\neg\mathfrak{P}\rightarrow\mathfrak{P}$, $\neg(\mathfrak{A}_{[\text{人}]}\rightarrow\mathfrak{B}_{[\text{人}]})\rightarrow(\mathfrak{A}_{[\text{人}]}\rightarrow\mathfrak{B}_{[\text{人}]})$, $\neg(\mathfrak{A}_{[\text{人}]}\wedge\mathfrak{B}_{[\text{人}]})\rightarrow(\mathfrak{A}_{[\text{人}]}\wedge\mathfrak{B}_{[\text{人}]})$, $\neg(\mathfrak{A}_{[\text{人}]}\vee\mathfrak{B}_{[\text{人}]})\rightarrow(\mathfrak{A}_{[\text{人}]}\vee\mathfrak{B}_{[\text{人}]})$, $\neg(x)\mathfrak{A}_{[\text{人}]}(x)\rightarrow(x)\mathfrak{A}_{[\text{人}]}(x)$, and $\neg(\exists x)\mathfrak{A}_{[\text{人}]}(x)\rightarrow(\exists x)\mathfrak{A}_{[\text{人}]}(x)$ in ***LM*** respectively, and $(\neg\mathfrak{A})_{[\text{人}]}$ can be defined by $\neg(\mathfrak{A}_{[\text{人}]})$.

3. Main theorem.

Theorem. $\vdash_{LD}\mathfrak{A}$ *if and only if* $\vdash_{LM}\mathfrak{A}_{[\text{人}]}$;
$\vdash_{LK}\mathfrak{A}$ *if and only if* $\vdash_{LJ}\mathfrak{A}_{[\text{人}]}$.

This theorem is derived from the following lemmas.

Lemma 1. $\vdash_{LD}\mathfrak{A}\equiv\mathfrak{A}_{[\text{人}]}$, *and also* $\vdash_{LK}\mathfrak{A}\equiv\mathfrak{A}_{[\text{人}]}$.

Proof. This can be proved recursively by definition of the transformation, because $((\mathfrak{A}\rightarrow\text{人})\rightarrow\mathfrak{A})\rightarrow\mathfrak{A}$ holds in ***LD***.

Hence, if $\vdash_{LM}\mathfrak{A}_{[\text{人}]}$, then $\vdash_{LD}\mathfrak{A}$, because ***LD*** is stronger than ***LM***. Also if $\vdash_{LJ}\mathfrak{A}_{[\text{人}]}$, then $\vdash_{LK}\mathfrak{A}$.

Lemma 2. *If* $\vdash_{LD}\mathfrak{A}$, *then* $\vdash_{LM}\mathfrak{A}_{[\text{人}]}$.

To prove this lemma, we shall formulate ***LM*** and ***LD*** in Gentzen's style. In ***LM*** its negation is defined by constant proposition 人, so the schemata for ***LM*** are the positive part of ***LJ***. ***LD*** is obtained from ***LM***, fortifying by the schema

$$ND\frac{\neg\mathfrak{A},\,\Gamma\vdash\mathfrak{A}}{\Gamma\vdash\mathfrak{A}}.$$

Lemma 2′. *From any proof of a formula* $\mathfrak{A}$ *in* ***LD***, *a proof of* $\mathfrak{A}_{[\text{人}]}$ *in* ***LM*** *is obtained by carrying out transformation "* $_{[\text{人}]}$ *" on every constituent of it, and by adding some more steps.*

Proof. The proof is accomplished by showing that for each schema of ***LD***, there is a deduction in ***LM*** from its transformed sequent above to its transformed sequent below.

(1) Beginning sequent. $\mathfrak{A}_{[\text{人}]}\vdash\mathfrak{A}_{[\text{人}]}$ is also a beginning sequent for ***LM***.

(2) Schemata for logical constants (except ***ND***). These deductions are obtained similarly for all logical constants, so we shall prove only for disjunction (***D***1 and ***D***2).

Remark. In deductions, we shall use the following items without special notice:

(i) For each $\mathfrak{C}$, $\mathfrak{C}_{[\text{人}]}$ is rewritten in the form $(\mathfrak{C}'\rightarrow\text{人})\rightarrow\mathfrak{C}'$.

(ii) $\Gamma_{[\text{人}]}$ stands for the sequence of formulas obtained from Γ by carrying out the transformation on every constituent in Γ.

(iii) The inversion theorem for implication.

(iv) The transformation does not change variable conditions.

(v) Schemata for structure.

***D*1.**

$$\dfrac{\dfrac{\Gamma_{[人]} \vdash \mathfrak{A}_{[人]}}{\Gamma_{[人]} \vdash \mathfrak{A}_{[人]} \vee \mathfrak{B}_{[人]}}}{\Gamma_{[人]} \vdash (\mathfrak{A} \vee \mathfrak{B})_{[人]}}$$

Similarly for other succedent rules.

***D*2.**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathfrak{A}_{[人]}, \Gamma_{[人]} \vdash \mathfrak{C}_{[人]} \quad \mathfrak{B}_{[人]}, \Gamma_{[人]} \vdash \mathfrak{C}_{[人]}}{\mathfrak{A}_{[人]} \vee \mathfrak{B}_{[人]}, \Gamma_{[人]} \vdash \mathfrak{C}_{[人]}} \quad \mathfrak{C}' \vdash \mathfrak{C}' \quad 人 \vdash 人}{\mathfrak{C}' \to 人, \mathfrak{A}_{[人]} \vee \mathfrak{B}_{[人]}, \Gamma_{[人]} \vdash 人}}{\mathfrak{C}' \to 人, \Gamma_{[人]} \vdash \mathfrak{A}_{[人]} \vee \mathfrak{B}_{[人]} \to 人} \quad \dfrac{\dfrac{\mathfrak{A}_{[人]}, \Gamma_{[人]} \vdash \mathfrak{C}_{[人]} \quad \mathfrak{B}_{[人]}, \Gamma_{[人]} \vdash \mathfrak{C}_{[人]}}{\mathfrak{A}_{[人]} \vee \mathfrak{B}_{[人]}, \Gamma_{[人]} \vdash \mathfrak{C}_{[人]}}}{\mathfrak{A}_{[人]} \vee \mathfrak{B}_{[人]}, \Gamma_{[人]}, \mathfrak{C}' \to 人 \vdash \mathfrak{C}'}}{\mathfrak{C}' \to 人, (\mathfrak{A}_{[人]} \vee \mathfrak{B}_{[人]} \to 人) \to \mathfrak{A}_{[人]} \vee \mathfrak{B}_{[人]}, \Gamma_{[人]} \vdash \mathfrak{C}'}}{(\mathfrak{A}_{[人]} \vee \mathfrak{B}_{[人]} \to 人) \to \mathfrak{A}_{[人]} \vee \mathfrak{B}_{[人]}, \Gamma_{[人]} \vdash (\mathfrak{C}' \to 人) \to \mathfrak{C}'}}{(\mathfrak{A} \vee \mathfrak{B})_{[人]}, \Gamma_{[人]} \vdash \mathfrak{C}_{[人]}}$$

Our tactics of this transformation of deduction would be understood nicely by reading the formal deduction from beneath, especially the last three steps. Other antecedent rules can be transformed into a deduction in ***LM*** having similar part in the last three steps.

(3) Schema ***ND***.

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathfrak{C}' \to 人 \vdash \mathfrak{C}' \to 人 \quad \mathfrak{C}' \vdash \mathfrak{C}'}{\mathfrak{C}' \to 人, (\mathfrak{C}' \to 人) \to \mathfrak{C}' \vdash \mathfrak{C}'} \quad \dfrac{\mathfrak{C}' \vdash \mathfrak{C}' \quad 人 \vdash 人}{\mathfrak{C}', \mathfrak{C}' \to 人 \vdash 人}}{\mathfrak{C}' \to 人, (\mathfrak{C}' \to 人) \to \mathfrak{C}' \vdash 人}}{\mathfrak{C}' \to 人 \vdash ((\mathfrak{C}' \to 人) \to \mathfrak{C}') \to 人} \quad \dfrac{(\neg \mathfrak{C})_{[人]}, \Gamma_{[人]} \vdash \mathfrak{C}_{[人]}}{((\mathfrak{C}' \to 人) \to \mathfrak{C}') \to 人, \mathfrak{C}' \to 人, \Gamma_{[人]} \vdash \mathfrak{C}'}}{\dfrac{\mathfrak{C}' \to 人, \Gamma_{[人]} \vdash \mathfrak{C}'}{\Gamma_{[人]} \vdash \mathfrak{C}_{[人]}}}$$

(4) Schemata for structure. Evident.

By (1)-(4), Lemma 2′ is proved. Therefore also Lemma 2.

Lemma 3. *If* $\vdash_{LK} \mathfrak{A}$, *then* $\vdash_{LJ} \mathfrak{A}_{[人]}$.

Proof. ***LK*** and ***LJ*** are obtained from ***LD*** and ***LM*** respectively by taking $\mathfrak{A}, \neg\mathfrak{A} \vdash \mathfrak{B}$, or $人 \vdash \mathfrak{B}$ as the added biginning sequent. Therefore we can conclude Lemma 3 from Lemma 2′.

Remark. When transformation "$_{[人]}$" is simplified as follows

$$(\mathfrak{A} \to \mathfrak{B})_{[人]} \rightleftharpoons \mathfrak{A}_{[人]} \to \mathfrak{B}_{[人]},$$
$$(\mathfrak{A} \wedge \mathfrak{B})_{[人]} \rightleftharpoons \mathfrak{A}_{[人]} \wedge \mathfrak{B}_{[人]},$$
$$((x)\mathfrak{A}(x))_{[人]} \rightleftharpoons (x)\mathfrak{A}_{[人]}(x),$$

and others are same as before, we can also obtain the same result by slightly complicated proofs.

4. Conclusion. We can see by the above theorem that the transformation gives a reduction of ***LD*** to ***LM*** and a reduction of

LK to ***LJ***, not to ***LM***. For, each transformed formula $\mathfrak{A}_{[\wedge]}$ is provable in ***LM*** if and only if $\mathfrak{A}$ is provable in ***LD*** which is weaker than ***LK***.

Now ***LD*** and ***LK*** are obtained from ***LM*** assuming further $(\neg\mathfrak{A}\rightarrow\mathfrak{A})\rightarrow\mathfrak{A}$ (*Clavius' principle*, equivalent to $\mathfrak{A}\vee\neg\mathfrak{A}$ *tertium non datur* on ***LM***) and $\neg\neg\mathfrak{A}\rightarrow\mathfrak{A}$ respectively. In the definition of transformations, [6] Kleene and [7] Kuroda (as [2] Glivenko for proposition logic) carried each subformula $\mathfrak{A}$ into $\neg\neg\mathfrak{A}$, the *first part* of $\neg\neg\mathfrak{A}\rightarrow\mathfrak{A}$, and have obtained in reality reductions of ***LK*** to ***LM***. On the other hand, in our case (as [1] Curry for proposition logic), the transformation "$_{[\wedge]}$" carries $\mathfrak{A}$ into $\neg\mathfrak{A}\rightarrow\mathfrak{A}$, the *first part* of $(\neg\mathfrak{A}\rightarrow\mathfrak{A})\rightarrow\mathfrak{A}$, which is weaker than $\neg\neg\mathfrak{A}\rightarrow\mathfrak{A}$ on ***LM***, and we obtain reductions of ***LD*** to ***LM*** and of ***LK*** to ***LJ*** (not to ***LM***).

In the above discussion we are searching for a reduction of ***LD*** to ***LM***. However, one could hope reductions of ***LD*** and others to weaker (than ***LM***) logic such as ***LP***, or ***LO*** in [8] Ono. In the paper [8] Ono investigates systematically the interpretations, generalization of reductions, of various kind of logics in ***LO*** by introducing a new symbol. Therefore if we superpose Ono's $\mathfrak{R}$-transformation on our transformation above, we can obtain an interpretation of ***LD*** in ***LO***.

References

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[8] Ono, K.: On universal character of the primitive logic. Nagoya Math. J., **27**, 331-353 (1966).

3) In [1] Curry, ***LD*** is originally formulated by adding the schema

$$\frac{\mathfrak{A},\ \Gamma\vdash\mathfrak{A}\rightarrow\mathfrak{A},\ \Gamma\vdash\mathfrak{B}}{\Gamma\vdash\mathfrak{B}}$$

to ***LM***, but the equivalence of these schemata on ***LM*** is easily shown.