195. On Free Abelian *m*-Groups. III

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In this part, the notion of tensor product of abelian m-groups will be introduced.

Definition. The *tensor product* of the abelian m-groups M and N is defined as F/θ and is denoted by $M\boxtimes N$.

If $|(x, y)|/\theta$ is denoted by $x\boxtimes y$, observe that

$$[x_1x_2\cdots x_m]\boxtimes y=[(x_1\boxtimes y)(x_2\boxtimes y)\cdots(x_m\boxtimes y)],$$
$$x\boxtimes[y_1y_2\cdots y_m]=[(x\boxtimes y_1)(x\boxtimes y_2)\cdots(x\boxtimes y_m)],$$

and $x^{\langle n\rangle}\boxtimes y=x\boxtimes y^{\langle n\rangle}=(x\boxtimes y)^{\langle n\rangle}$.

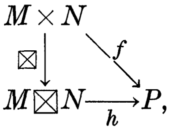
Theorem 9. *Let M, N, P be arbitrary abelian m-groups and $f: M\times N\to P$ be a function satisfying the conditions*

(a) $f([x_1x_2\cdots x_m], y)=[f(x_1, y)f(x_2, y)\cdots f(x_m, y)]$,

(b) $f(x, [y_1y_2\cdots y_m])=[f(x, y_1)f(x, y_2)\cdots f(x, y_m)]$,

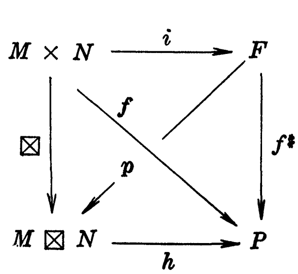
(c) $f(x^{\langle n\rangle}, y)=f(x, y^{\langle n\rangle})$,

for all $x, x_1, \cdots, x_m\in M$ and $y, y_1, \cdots, y_m\in N$. Then there exists uniquely an m-group homomorphism $h: M\boxtimes N\to P$ such that the following diagram is commutative



that is, $h(x\boxtimes y)=f(x, y)$ for all $x\in M$ and $y\in N$.

Proof. Let F be the free abelian m-group on $M\times N$ and $i: M\times N\to F$ be the injection $i(x, y)=|(x, y)|$. Consider the following diagram.



By Theorem 4, f possesses a unique homomorphic extension $f^{\#}: F\to P$ such that $f^{\#}\cdot i(x, y)=f(x, y)$ so that $f^{\#}(|(x, y)|)=f(x, y)$. Since

$$f^{\#}(|([x_1x_2\cdots x_m], y)|)=f([x_1x_2\cdots x_m], y)$$
$$=[f(x_1, y)f(x_2, y)\cdots f(x_m, y)]=[f^{\#}(|(x_1, y)|)\cdots f^{\#}(|(x_m, y)|)],$$

$$f^{\#}(|(x,[y_1y_2\cdots y_m])|)=f(x,[y_1y_2\cdots y_m])$$
$$=[f(x,y_1)f(x,y_2)\cdots f(x,y_m)]=[f^{\#}(|(x,y_1)|)\cdots f^{\#}(|(x,y_m)|)],$$
$$f^{\#}(|(x^{\langle n\rangle},y)|)=f(x^{\langle n\rangle},y)=f(x,y^{\langle n\rangle})=f^{\#}(|(x,y^{\langle n\rangle})|),$$

then $\theta\subseteqq f^{\#}\circ(f^{\#})^{-1}$. This implies then that $f^{\#}$ factors through the natural homomorphism $p\colon F\to F/\theta=M\boxtimes N$, that is to say, there exists a homomorphism $h\colon M\boxtimes N\to P$ such that $h\circ p=f^{\#}$. Thus

$$h(x\boxtimes y)=h(p(|(x,y)|))=(h\circ p)(|(x,y)|)=f^{\#}(|(x,y)|)=f(x,y).$$

The proof is thus completed.

The following follows from the preceding theorem and its proof is similar to the proof in ordinary groups.

Theorem 10. (a) $M\boxtimes N\cong N\boxtimes M$;

(b) $(M\boxtimes N)\boxtimes P\cong M\boxtimes(N\boxtimes P)$,

for any three abelian m-groups M, N, and P.

The following Lemmata will be needed in the following.

Lemma A. (1) *If $x_1,\cdots,x_{m-1}\in M$ such that $(x_1,\cdots,x_{m-1})$ is an $(m-1)$-adic identity of M and $y\in N$, then $((x_1\boxtimes y),\cdots,(x_m\boxtimes y))$ is an $(m-1)$-adic identity of $M\boxtimes N$.*

(2) *If $y_1,\cdots,y_{m-1}\in N$ such that $(y_1,\cdots,y_{m-1})$ is an $(m-1)$-adic identity of N and $x\in M$, then $((x\boxtimes y_1),\cdots,(x\boxtimes y_m))$ is an $(m-1)$-adic identity of $M\boxtimes N$.*

Proof. For each $x\in M$, note that $[(x_1\boxtimes y)\cdots(x_{m-1}\boxtimes y)(x\boxtimes y)]=[x_1x_2\cdots x_{m-1}x]\boxtimes y=x\boxtimes y$. Similarly $[(x\boxtimes y)(x_1\boxtimes y)\cdots(x_{m-1}\boxtimes y)]=x\boxtimes y$. The proof of (2) is analogous.

Lemma B. (1) *If $x_1,\cdots,x_{m-1}\in M$ such that $(x_1,\cdots,x_{m-1})$ is an $(m-1)$-adic identity of M and $y_1,\cdots,y_s\in N$, then $((x_1\boxtimes y_1),\cdots,(x_{m-1}\boxtimes y_s))$ is an $(m-1)s$-adic identity of $M\boxtimes N$.*

(2) *If $y_1,\cdots,y_{m-1}\in N$ such that $(y_1,\cdots,y_{m-1})$ is an $(m-1)$-adic identity of N and $x_1,\cdots,x_r\in M$, then $((x_1\boxtimes y_1),\cdots,(x_r\boxtimes y_{m-1}))$ is an $r(m-1)$-adic identity of $M\boxtimes N$.*

Proof. We shall only prove (1) since the proof of (2) is similar. Let $y_{s+1},\cdots,y_{m-1}\in N$ such that $(y_1,\cdots,y_{m-1})$ is and $(m-1)$-adic identity of N; then

$$[(x_1\boxtimes y_1)\cdots(x_{m-1}\boxtimes y_s)(x\boxtimes y)]=[(x_1\boxtimes y_1)\cdots(x_{m-1}\boxtimes y_s)[(x\boxtimes y_1)\cdots$$
$$(x\boxtimes y_{m-1})(x\boxtimes y)]]=[[(x_1\boxtimes y_1)\cdots(x_{m-1}\boxtimes y_1)(x\boxtimes y_1)]\cdots$$
$$[(x_1\boxtimes y_s)\cdots(x_{m-1}\boxtimes y_s)(x\boxtimes y_s)](x\boxtimes y_{s+1})\cdots(x\boxtimes y_{m-1})(x\boxtimes y)]$$
$$=[(x\boxtimes y_1)\cdots(x\boxtimes y_s)(x\boxtimes y_{s+1})\cdots(x\boxtimes y_{m-1})(x\boxtimes y)]=x\boxtimes y.$$

Lemma C. (1) *If $(x_1,\cdots,x_r)\overset{r}{\sim}(x_1',\cdots,x_r')$ in M with $r\leqq m-1$ and $y_1,\cdots,y_s\in N$, then $((x_1\boxtimes y_1),\cdots,(x_r\boxtimes y_s))\overset{rs}{\sim}((x_1'\boxtimes y_1),\cdots,(x_r'\boxtimes y_s))$ in $M\boxtimes N$.*

(2) *If $(y_1,\cdots,y_s)\overset{s}{\sim}(y_1',\cdots,y_s')$ in N with $s\leqq m-1$ and $x_1,\cdots,x_r\in M$, then $((x_1\boxtimes y_1),\cdots,(x_r\boxtimes y_s))\overset{rs}{\sim}((x_1\boxtimes y_1'),\cdots,(x_r\boxtimes y_s'))$*

in $M\boxtimes N$.

Proof. Suppose $r \leqq m-2$. Let $x_{r+1}, \cdots, x_{m-1} \in M$ such that $(x_1, \cdots, x_{m-1})$ and hence also $(x'_1, \cdots, x'_r, x_{r+1}, \cdots, x_{m-1})$ is an $(m-1)$-adicidentity of M. For each $x \in M$, we have by Lemma B(1),

$$[(x_1\boxtimes y_1) \cdots (x_r\boxtimes y_s)(x_{r+1}\boxtimes y_1) \cdots (x_{m-1}\boxtimes y_s)(x\boxtimes y)] = x\boxtimes y$$
$$= [(x'_1\boxtimes y_1) \cdots (x'_r\boxtimes y_s)(x_{r+1}\boxtimes y_1) \cdots (x_{m-1}\boxtimes y_s)(x\boxtimes y)].$$

Whence the result follows. The proof for the case when $r=m-1$ is analogous. The proof of the second part will be omotted, since it is the same as the preceding.

Lemma D. *If* $(x_1, \cdots, x_r)\overset{r}{\sim}(x'_1, \cdots, x'_r)$ *in* M *and* $(y_1, \cdots, y_s)$ $\overset{s}{\sim}(y'_1, \cdots, y'_s)$ *in* N, *then*

$$(x_1\boxtimes y_1, \cdots, x_r\boxtimes y_s)\overset{rs}{\sim}(x'_1\boxtimes y'_1, \cdots, x'_r\boxtimes y'_s).$$

Proof. By Lemma C(1) we have in $M\boxtimes N$,

$$(x_1\boxtimes y_1, \cdots, x_r\boxtimes y_s)\overset{rs}{\sim}(x'_1\boxtimes y_1, \cdots, x'_r\boxtimes y_s);$$

by Lemma C(2) we have in $M\boxtimes N$,

$$(x'_1\boxtimes y_1, \cdots, x'_r\boxtimes y_s)\overset{rs}{\sim}(x'_1\boxtimes y'_1, \cdots, x'_r\boxtimes y'_s).$$

The final result follows by transitivity.

Lemma E. *Let* $G = M \cup M^2 \cup \cdots \cup M^{m-1}$ *be the containing group of the m-group* M. *Then* $(x_1, \cdots, x_i)\overset{i}{\sim}(x'_1, \cdots, x'_i)$ *in* M *if and only if* $x_1x_2 \cdots x_i = x'_1x'_2 \cdots x'_i$ *in* G.

The proof of Lemma E is clear.

Theorem 11. *The tensor product of two abelian m-groups is a coset of the tensor product of their respective containing groups (by the Post Coset Theorem).*

Proof. Let M and N be two abelian m-groups and $M\boxtimes N$ be their tensor product. Denote respectively by $A = M \cup 2M \cup \cdots \cup (m-1)M$, $B = N \cup 2N \cup \cdots \cup (m-1)N$, and $C = (M\boxtimes N) \cup 2(M\boxtimes N) \cup \cdots \cup (m-1)(M\boxtimes N)$ their containg (abelian) groups by the Post Coset Theorem. Define $f\colon A \times B \to C$ such that

$$f\Big(\sum_{i=1}^{r} x_i, \sum_{j=1}^{s} y_j\Big) = \sum_{i=1}^{r}\sum_{j=1}^{s}(x_i\boxtimes y_j)$$

for all $x_1, \cdots, x_r \in$ and $y_1, \cdots, y_s \in N$. Note that

$$f\Big(\sum_{i=1}^{r} x_i, \sum_{j=1}^{s} y_j\Big) \in (M\boxtimes N)^{m-1}$$

if $rs \equiv 0 \pmod{m-1}$; otherwise $f\Big(\sum_{i=1}^{r} x_i, \sum_{j=1}^{s} y_j\Big) \in (M\boxtimes N)^t$, where t is the residue of $rs \,(\mathrm{mod}\, m-1)$. By the preceding Lemmata D and E, f is clearly well-defined. Moreover,

$$f\Big(\sum_{i=1}^{r} x_i + \sum_{i=r+1}^{n} x_i, \sum_{j=1}^{s} y_j\Big) = f\Big(\sum_{i=1}^{n} x_i, \sum_{j=1}^{s} y_j\Big) = \sum_{i=1}^{n}\sum_{j=1}^{s}(x_i\boxtimes y_j)$$
$$= \sum_{i=1}^{r}\sum_{j=1}^{s}(x_i\boxtimes y_j) + \sum_{i=r+1}^{n}\sum_{j=1}^{s}(x_i\boxtimes y_j) = f\Big(\sum_{i=1}^{r} x_i, \sum_{j=1}^{s} y_j\Big) + f\Big(\sum_{i=r+1}^{n} x_i, \sum_{j=1}^{s} y_j\Big).$$

Similarly, $f\left(\sum_{i=1}^{r} x_i,\ \sum_{j=1}^{s} y_j+\sum_{j=s+1}^{n} y_j\right)=f\left(\sum_{i=1}^{r} x_i,\ \sum_{j=1}^{s} y_j\right)+f\left(\sum_{i=1}^{r} x_i,\ \sum_{j=s+1}^{n} y_j\right)$. Now, note that if $x_1, \cdots, x_{m-1} \in M$ such that $(x_1, \cdots, x_{m-1})$ is an $(m-1)$-adic identity of M, i.e. $\sum_{i=1}^{m-1} x_i=0$ in A, then $\sum_{i=r+1}^{m-1} x_i=-\sum_{i=1}^{r} x_i$ in A. Thus, $f\left(-\sum_{i=1}^{r} x_i,\ \sum_{j=1}^{s} y_j\right)=f\left(\sum_{i=r+1}^{m-1} x_i,\ \sum_{j=1}^{s} y_j\right)\sum_{i=r+1}^{m-1}\sum_{j=1}^{s}(x_i \boxtimes y_j)$. By Lemmata B and E, then $\sum_{i=1}^{m-1}\sum_{j=1}^{s}(x_i \boxtimes y_j)=0$ in $M \boxtimes N$, and hence

$$\sum_{i=r+1}^{m-1}\sum_{j=1}^{s}(x_i \boxtimes y_j)=-\sum_{i=1}^{r}\sum_{j=1}^{s}(x_i \boxtimes y_j).$$

Similarly,

$$f\left(\sum_{i=1}^{r} x_i,\ -\sum_{j=1}^{s} y_j\right)=f\left(\sum_{i=1}^{r} x_i,\ \sum_{j=s+1}^{m-1} y_j\right)=\sum_{i=1}^{r}\sum_{j=s+1}^{m-1}(x_i \boxtimes y_j)=-\sum_{i=1}^{r}\sum_{j=1}^{s}(x_i \boxtimes y_j).$$

whence

$$f\left(-\sum_{i=1}^{r} x_i,\ \sum_{j=1}^{s} y_j\right)=f\left(\sum_{i=1}^{r} x_i,\ -\sum_{j=1}^{s} y_j\right).$$

Hence, by the analogue Theorem 9 for abelian groups, f extends to a homomorphism $f^{\#}\colon A \otimes B \to C$ such that

$$f^{\#}\left(\sum_{i=1}^{r} x_i \otimes \sum_{j=1}^{s} y_j\right)=\sum_{i=1}^{r}\sum_{j=1}^{s}(x_i \boxtimes y_j).$$

This is obviously an epimorphism. To show that $f^{\#}$ is also a monomorphism, suppose

$$f^{\#}\left(\sum_{k=1}^{n}\left(\sum_{i=1}^{r_k} x_i^k \otimes \sum_{j=1}^{s_k} y_j^k\right)\right)=f^{\#}\left(\sum_{k=1}^{n'}\left(\sum_{i=1}^{r'_k} x_i'^k \otimes \sum_{j=1}^{s'_k} y_j'^k\right)\right)$$

so that $\sum_{k=1}^{n}\sum_{i=1}^{r_k}\sum_{j=1}^{s_k}(x_i^k \boxtimes y_j^k)=\sum_{k=1}^{n'}\sum_{i=1}^{r'_k}\sum_{j=1}^{s'_k}(x_i'^k \boxtimes y_j'^k)$. However, since $\theta \subseteqq \theta^*$, then

$$\sum_{k=1}^{n}\sum_{i=1}^{r_k}\sum_{j=1}^{s_k}(x_i^k \otimes y_j^k)=\sum_{k=1}^{n'}\sum_{i=1}^{r'_k}\sum_{j=1}^{s'_k}(x_i'^k \otimes y_j'^k)$$

or

$$\sum_{k=1}^{n}\left(\sum_{i=1}^{r_k} x_i^k \otimes \sum_{j=1}^{s_k} y_j^k\right)=\sum_{k=1}^{n'}\left(\sum_{i=1}^{r'_k} x_i'^k \otimes \sum_{j=1}^{s'_k} y_i'^k\right).$$

Whence $f^{\#}$ is an isomorphism.

Theorem 12. *$Z \boxtimes M=M$, where M is an arbitrary abelian m-group and Z is the infinite cyclic m-group of integers.*

Proof. Consider the function $f\colon Z \times M \to M$ such that $f(n, x)=x^{\langle n\rangle}$. Then f satisfies the following conditions:

(1) $f([n_1 n_2 \cdots n_m], x)=x^{\langle [n_1 n_2 \cdots n_m]\rangle}=x^{\langle n_1+n_2+\cdots+n_m+1\rangle}[x^{\langle n_1\rangle}x^{\langle n_2\rangle} \cdots x^{\langle n_m\rangle}]=[f(n_1, x)f(n_2, x) \cdots f(n_m, x)]$,

(2) $f(n, [x_1 x_2 \cdots x_m])=[x_1 x_2 \cdots x_m]^{\langle n\rangle}=[x_1^{\langle n\rangle} x_2^{\langle n\rangle} \cdots x_m^{\langle n\rangle}]=[f(n, x_1)f(n, x_2) \cdots f(n, x_m)]$,

(3) $f(n^{\langle k\rangle}, x)=f(kn(m-1)+k+n, x)=x^{\langle kn(m-1)+k+n\rangle}=(x^{\langle k\rangle})^{\langle n\rangle}=f(n, x^{\langle k\rangle})$. Thus, f extends to a homomorphism $f^{\#}\colon Z \boxtimes M \to M$ such that

$f^{\#}(n\boxtimes x)=x^{\langle n\rangle}$ or more generally, $f^{\#}\left(\sum_i (n_i\boxtimes x_i)^{\langle k_i\rangle}\right)=\sum_i (x_i^{\langle n_i\rangle})^{\langle k_i\rangle}$.

Define $g^{\#}: M\rightarrow Z\boxtimes M$ such that $g^{\#}(x)=0\boxtimes x$. Since

$$\begin{aligned} f^{\#}([x_1x_2\cdots x_m]) &= 0\boxtimes[x_1x_2\cdots x_m] \\ &= [(0\boxtimes x_1)(0\boxtimes x_2)\cdots(0\boxtimes x_m)]=[g^{\#}(x_1)g^{\#}(x_2)\cdots g^{\#}(x_m)], \end{aligned}$$

then $g^{\#}$ is an m-group homomorphism. Now, observe that

$$f^{\#}g^{\#}(x)=f^{\#}(0\boxtimes x)=x^{\langle 0\rangle}=x=1_M(x)$$

and

$$g^{\#}f^{\#}\left(\sum_i (n_i\boxtimes x_i)^{\langle k_i\rangle}\right)=g^{\#}\left(\sum_i (x_i^{\langle n_i\rangle})^{\langle k_i\rangle}\right)=0\boxtimes\sum_i (x_i^{\langle n_i\rangle})^{\langle k_i\rangle}=\sum_i(0\boxtimes(x_i^{\langle n_i\rangle})^{\langle k_i\rangle})$$

$$=\sum_i (0\boxtimes x_i^{\langle n_i\rangle})^{\langle k_i\rangle}=\sum_i (0^{\langle n_i\rangle}\boxtimes x_i)^{\langle k_i\rangle}=\sum_i (n_i\boxtimes x_i)^{\langle k_i\rangle}=1_{Z\boxtimes M}\left(\sum_i (n_i\boxtimes x_i)^{\langle k_i\rangle}\right).$$

Whence, $f^{\#}$ and $g^{\#}$ are isomorphisms inverse to each other.

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