# 187. A Note on Almost-Countably Paracompact Spaces

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In a recent paper [1] the concept of almost-countably paracompact spaces has been introduced. A space $X$ is said to be almost-countably paracompact if for each countable open covering $\mathcal{A}$ of $X$ there exists a locally-finite family $\mathcal{B}$ of open subsets of $X$ which refines $\mathcal{A}$ and the family of closures of members of $\mathcal{B}$ forms a covering of $X$. In the present note we give a characterization of such spaces.

$A^\circ$ denotes the interior of $A$ and $\bar{A}$ denotes the closure of $A$.

**Theorem 1.** *For a topological space $(X, \zeta)$ the following are equivalent:*

(a) *$X$ is almost-countably paracompact.*

(b) *For every decreasing sequence $\{F_i\}$ of closed subsets of $X$ such that $F_i^0 \neq \phi$ for all $i$ and $\bigcap_{i \in N} F_i \subset U$ where $U$ is open, there exists a decreasing sequence $\{G_i\}$ of open subsets of $X$ such that $F_i^0 \subset \bar{G}_i$ for each $i$ and $\bigcap_{i \in N} \bar{G}_i \subset \bar{U}$.*

(c) *For each decreasing sequence $\{F_i\}$ of closed subsets of $X$ such that $F_i^0 \neq \phi$ for each $i$ and $\bigcap_{i \in N} F_i \subset U$ where $U$ is open, there exists a decreasing sequence $\{H_i\}$ of closed subsets of $X$ such that $F_i^0 \subset H_i$ for each $i$ and $\bigcap_{i \in N} H_i \subset \bar{U}$.*

**Proof.** $(a) \Rightarrow (b)$. Since $\bigcap_{i \in N} F_i \subset U$, therefore $X \sim U \subset X \sim \bigcap_{i \in N} F_i = \bigcup_{i \in N} X \sim F_i$. Thus $\{X \sim F_i : i \in N\} \cup \{U\}$ is a countable open covering of $X$. Therefore there exists a locally-finite family $\{V_i\}$ of open subsets of $X$ such that $V_i \subset X \sim F_i$ for each $i$ and $(\bigcup_{i \in N} \bar{V}_i) \cup \bar{U} = X$. For each $i$ let $G_i = \bigcup_{n=i+1}^{\infty} (V_n \cup U)$. Then $\bar{G}_i \supset X \sim (\bar{V}_1 \cup \cdots \cup \bar{V}_i) \supset X \sim \overline{X \sim F_i} = F_i^0$. Thus $\{G_i\}$ is a decreasing sequence of open sets such that $F_i^0 \subset \bar{G}_i$. We shall show now that $\bigcap_{i \in N} \bar{G}_i \subset \bar{U}$. If a point $x \in \bigcap_{i \in N} \bar{G}_i$ and $x \notin \bar{U}$, then $\{\bar{V}_i\}$ being locally-finite there exists an open set $M_x$ which intersects finitely many sets $\bar{V}_i^s$. Therefore there exists an integer $i$ such that $M_x \cap \left( \bigcup_{n=i+1}^{\infty} \bar{V}_n \right) = \phi$. Therefore $x \notin \bigcup_{n=i+1}^{\infty} \bar{V}_n$. Also, $x \notin \bar{U}$. Therefore $x \notin \bar{G}_i$, which is a contradiction. Therefore $x \in \bigcap_{i \in N} \bar{G}_i$, which implies $x \in \bar{U}$. Thus $\bigcap_{i \in N} \bar{G}_i \subset \bar{U}$.

$(b) \Rightarrow (c)$. This is obvious since we can take $H_i = \bar{G}_i$.

$(c) \Rightarrow (a)$. Let $\{U_i\}$ be any countable open covering of $X$. For each $i \in N$, let $F_i = X \sim \bigcup_{n=1}^{i} U_n$. Then we can assume that $F_i^0 \neq \phi$ for each $i$ because if $F_i^0 = \phi$ for some $i$ then $\bigcup_{n=1}^{i} \bar{U}_n = X$ and the space is almost-countably paracompact. Thus $\{F_i\}$ is a decreasing sequence of closed sets such that $F_i^0 \neq \phi$ and $\bigcap_{i \in N} F_i = \phi$. Now $\phi$ is open and therefore by hypothesis there exists a decreasing sequence $\{H_i\}$ of closed sets such that $F_i^0 \subset H_i$ for each $i$ and $\bigcap_{i \in N} H_i \subset \bar{\phi}$, i.e., $\bigcap_{i \in N} H_i = \phi$. For each $i$ let $E_i = X \sim H_i$ and let $V_i = U_i - \bar{E}_{i-1}$, $V_1 = U_1$. Then $\{V_i\}$ is a family of open sets such that $V_i \subset U_i$ for each $i$. Let $p$ be any point in $X$ and let $i$ be the first integer such that $p \in \bar{U}_i$. Then $p \in \bar{U}_i \sim \bigcup_{n=1}^{i-1} \bar{U}_n$. Also, $E_i \subset X \sim F_i^0 = \bigcup_{n=1}^{i} \bar{U}_n$. Now,

$$\bar{V}_i = \overline{U_i \sim \bar{E}_{i-1}} = \overline{U_i \cap (X \sim \bar{E}_{i-1})} = \overline{\bar{U}_i \cap (X \sim \bar{E}_{i-1})}$$

(because $X \sim \bar{E}_{i-1}$ is an open set and for any open set $O$ and any set $A$ we have $\overline{O \cap \bar{A}} = \overline{O \cap A}$). Also, because $\bar{E}_{i-1} \subset \bigcup_{n=1}^{i-1} \bar{U}_n$, therefore $X \sim \bar{E}_{i-1} \supset X \sim \bigcup_{n=1}^{i-1} \bar{U}_n$. Thus

$$\bar{V}_i = \overline{\bar{U}_i \cap (X \sim \bar{E}_{i-1})} \supset \overline{\bar{U}_i \cap (X \sim \bigcup_{n=1}^{i-1} \bar{U}_n)} = \overline{\bar{U}_i \sim \bigcup_{n=1}^{i-1} \bar{U}_n}.$$

Therefore $p \in \bar{V}_i$. Thus $\bigcup_{i \in N} \bar{V}_i = X$. Also, $\{V_i\}$ is locally-finite, for if $x \in X$, then since $\bigcap_{i \in N} H_i = \phi$ therefore $\bigcup_{i \in N} X \sim H_i = X$, i.e., $\bigcup_{i \in N} E_i = X$. Therefore $x \in E_k$ for some $k \in N$. Also, $E_k \cap V_i = \phi$ for all $i > k$. Thus $E_k$ is an open set containing $x$ which intersects finitely many members of $\{V_i\}$ and therefore $\{V_i\}$ is locally-finite. Hence $X$ is almost-countably paracompact.

**Theorem 2.** *A sufficient condition for a space $X$ to be almost-countably paracompact is that for every decreasing sequence $\{A_i\}$ of regularly-open sets such that $\bigcap_{i \in N} A_i = \phi$ there exists a decreasing sequence $\{G_i\}$ of open subsets of $X$ such that $\bigcap_{i \in N} \bar{G}_i = \phi$ and $\bar{G}_i \supset F_i$ for each $i$.*

**Proof.** Let $\{U_i\}$ be any countable open covering of $X$. For each $i$ let $F_i = X \sim \bigcup_{n=1}^{i} \bar{U}_n$. Then $\{F_i\}$ is a decreasing sequence of regularly-open sets such that $\bigcap_{i \in N} F_i = \phi$. Then there exists a decreasing sequence $\{G_i\}$ of open subsets of $X$ such that $F_i \subset \bar{G}_i$ and $\bigcap_{i \in N} \bar{G}_i = \phi$. Now put $\bar{G}_i = H_i$. The rest of the proof is exactly as that of implication $(c) \Rightarrow (a)$ of Theorem 1.

## Reference

[1] M. K. Singal and Shashi Prabha Arya: On $m$-paracompact spaces (to appear).