# 186. On the Representations of SL(3, C). I

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1. We shall determine the intertwinning operators and the equivalence relation among the representations of the group $SL(3, C)$, generalizing the method described in [1] for $SL(2, C)$. We denote by $G$ the group $SL(3, C)$ and we adopt the notations of the book [2] thoughout this paper, but elements of $Z$ will be denoted by

$$z=\begin{bmatrix}1 & & \\ z_1 & 1 & \\ z_3 & z_2 & 1\end{bmatrix}, \quad \text{especially} \quad z_1=\begin{bmatrix}1 & & \\ z_1 & 1 & \\ & & 1\end{bmatrix}$$

and so on. Let $W$ be the Weyl group of $G$ consisted of $s_0=e$, $s_1$, $s_2$, $s_3=s_2s_1$, $s_4=s_1s_2$ and $s_5=s_1s_2s_1=s_2s_1s_2$, where

$$s_1=\begin{bmatrix} & 1 & \\ 1 & & \\ & & -1\end{bmatrix}, \quad s_2=\begin{bmatrix}-1 & & \\ & & 1 \\ & 1 & \end{bmatrix}.$$

Let $G^0$ be the set of all $g$ such that $g_{33}\cdot g^{11}\neq 0$, then $g=kz$ for all $g\in G^0$.

2. Let $\chi$ be an integral character of $D$: $\chi(\delta)=(\delta_2\delta_3)^{(l_1,m_1)}\delta_3^{(l_2,m_2)}$ ($l_k, m_k>0$), and $\mathcal{E}_\chi$ be the finite dimensional vector space of polynomials $\varphi$ on $Z$ which are at most of degree $(l_1-1, m_1-1)$ with respect to $z_1, z_1z_2-z_3$ and of degree $(l_2-1, m_2-1)$ with respect to $z_2, z_3$. Then, according to the theorem of Cartan and Weyl, for every finite dimensional irreducible representation of $G$ there exists $\chi$ such that given representation $E^\chi$ is realized on $\mathcal{E}_\chi$ by $E_g^\chi\varphi(z)=\chi\beta^{-1/2}(k_g)\varphi(z_g)$.

Now let $\chi=(\lambda, \mu)$ be a complex character of $D$: $\chi(\delta)=(\delta_2\delta_3)^{(\lambda_1,\mu_1)}\delta_3^{(\lambda_2,\mu_2)}$ ($\lambda_k$, $\mu_k$ are complex numbers and $\lambda_k-\mu_k$ are integers), then we can construct a representation $\{T^\chi, \mathcal{D}_\chi\}$ as follows. Let $\mathcal{D}_\chi$ be the vector space of $C^\infty$-functions $\varphi$ on $Z$, satisfying the condition that for every $s\in W$ $\varphi_s(z)=\chi\beta^{-1/2}(k_s)\varphi(z_s)$ is also a $C^\infty$-function. The topology of $\mathcal{D}_\chi$ is defined by the compact uniform convergence of every derivative for every $\varphi_s$ ($s\in W$). The operator $T_g^\chi$ on $\mathcal{D}_\chi$ is defined by $T_g^\chi\varphi(z)=\chi\beta^{-1/2}(k_g)\varphi(z_g)$. This representation is identical with the induced representation $T^\chi=Ind\{\chi\,|\,K\rightarrow G\}$. If all $\lambda_k$, $\mu_k$ are positive integers, the representation $\{E^\chi, \mathcal{E}_\chi\}$ is contained in $\{T^\chi, \mathcal{D}_\chi\}$ as a sub-representation.

3. Let $B(\varphi, \psi)$ be a continuous bilinear form on $\mathcal{D}_\chi\times\mathcal{D}_{\chi'}$ such

that $B(T_g^{\chi}\varphi, T_g^{\chi'}\psi)=B(\varphi, \psi)$. We denote by $C_0^\infty$ the totallity of $C^\infty$-functions on $G$ with compact support and define the continuous linear mapping of $C_0^\infty$ onto $\mathscr{D}_\chi$ as follows:

$$\pi^{\chi}(f)=\int f(kz)\chi^{-1}\beta^{1/2}(k)dk.$$

Then we can obtain a continuous bilinear form $B_1$ on $C_0^\infty \times C_0^\infty$ such that $B_1(f, h)=B(\pi^{\chi}(f), \pi^{\chi'}(h))$. We have

$$B_1(f, h)=\int f(g_1g_2)h(g_2)dT(g_1)dg_2,$$

where $dT$ is a distribution on $G$ which satisfies

$$(G) \qquad dT(k_1^{-1}gk_2)=\chi\beta^{1/2}(k_1)\chi'\beta^{1/2}(k_2)dT(g).$$

To obtain all invariant bilinear forms is equivalent to the problem to obtain $dT$ satisfying the condition $(G)$.

4. It is sufficient to consider $dT$ on each $KsK$ in order to obtain $dT$, since the condition $(G)$ is given on the $K$-$K$ double cosets and $G=\sum KsK$ $(s \in W)$.

(i) $Ks_5K=G^0s_5$ is a dense open submanifold of $G$;

(ii a) $Ks_3K$ and (ii b) $Ks_4K$ are seven-dimensional submanifolds of $G$ and their union is dense open in $G-Ks_5K$. They are contained in the boundary of $Ks_5K$;

(iii a) $Ks_1K$ and (iii b) $Ks_2K$ are six-dimensional and their union is dense open in the remaining part of $G$. They are contained in the boundary of the union of the above manifolds;

(iv) $Ks_0K=K$.

From the condition $(G)$, we can get the explicit form of the restriction $dT_5$ of $dT$ to $Ks_5K$. Then in order to determine $dT$ completely, it is sufficient to determine the extension $dT_5'$ to $G$ of $dT_5$ and to restrict $dT-dT_5'$ on $Ks_3K$ and to proceed analogously. With this method we arrive at the following results.

5. Corresponding to the cases enumerated in 4, we obtain the invariant bilinear forms $B(\varphi, \psi)$ in the following form.

(i) When and only when $\chi^{s_5}\chi'(\delta)=1$ $(\chi^s(\delta)=\chi(s\delta s^{-1}))$ and neither of pairs $(\lambda_k, \mu_k)$ $(k=1,2)$ is a pair of positive integers, $B(\varphi, \psi)$ exists and has the form

$$\int(z_1z_2-z_3)^{(-\lambda_1-1,-\mu_1-1)}\, z_3^{(-\lambda_2-1,-\mu_2-1)}\,\varphi(z\,z')\psi(z')dz\,dz';$$

(ii a) When and only when $\chi^{s_3}\chi'(\delta)=(\delta_1^2\delta_2)^{(i_1,j_1)}$, $i_1=\lambda_1$ or $0$, $j_1=\mu_1$ or $0$, and $(\lambda_2, \mu_2)$ is not a pair of positive integers,

$$\int z_1^{(-\lambda_1-\lambda_2+i_1-1,-\mu_1-\mu_2-j_1-1)}\, z_2^{(-\lambda_2-i_1-1,-\mu_2-j_1-1)}[(\partial/\partial z_1)^{(i_1,j_1)}\varphi](z_2z_1z')\psi(z')\,dz_1dz_2dz';$$

(ii b) When and only when $\chi^{s_4}\chi'(\delta)=(\delta_2^2\delta_3)^{(i_2,j_2)}$, $i_2=\lambda_2$ or $0$, $j_2=\mu_2$ or $0$, and $(\lambda_1, \mu_1)$ is not a pair of positive integers,

$$\int z_1^{(-\lambda_1-i_2-1,-\mu_1-j_2-1)} z_2^{(-\lambda_1-\lambda_2+i_2-1,-\mu_1-\mu_2+j_2-1)}$$
$$\times [(\partial/\partial z_2+z_1\partial/\partial z_3)^{(i_2,j_2)}\varphi](z_1z_2z')\psi(z')dz_1dz_2dz';$$

(iii a) When and only when $\chi^{s_1}\chi'(\delta)=(\delta_2\delta_3^2)^{(i_2,j_2)}(\delta_1\delta_3^2)^{(i_3,j_3)}$, $i_2=\lambda_2$ or $0, j_2=\mu_2$ or 0 and if $i_2=\lambda_2$, $i_3=\lambda_1$ or 0, if $i_2=0, i_3=\lambda_1+\lambda_2$ or 0 and if $j_2=\mu_2, j_3=\mu_1$ or 0, if $j_2=0, j_3=\mu_1+\mu_2$ or 0, and $(\lambda_1, \mu_1)$ is not a pair of positive integers,

$$\int z_1^{(-\lambda_2-i_1-1,-\mu_2-j_1-1)}[(\partial/\partial z_2)^{(i_3,j_3)}(\partial/\partial z_2+z_1\partial/\partial z_3)^{(i_2,j_2)}\varphi](z_1z')\psi(z')dz_1dz';$$

(iii b) When and only when $\chi^{s_2}\chi'(\delta)=(\delta_2\delta_3^2)^{(i_1,j_1)}(\delta_2^2\delta_3)^{(i_3,j_3)}$, $i_1=\lambda_1$ or 0, $j_1=\mu_1$ or 0 and if $i_1=\lambda_1, i_3=\lambda_2$ or 0, if $i_1=0, i_3=\lambda_1+\lambda_2$ or 0 and if $j_1=\mu_1, j_3=\mu_2$ or 0, if $j_1=0, j_3=\mu_1+\mu_2$ or 0, and $(\lambda_2, \mu_2)$ is not a pair of positive integers,

$$\int z_2^{(-\lambda_2-i_1-1,-\mu_2-j_1-1)}[(\partial/\partial z_1)^{(i_1,j_1)}(\partial/\partial z_1+z_2\partial/\partial z_3)^{(i_3,j_3)}\varphi](z_2z')\psi(z')dz_2dz'.$$

(iv) For $\chi\chi'(\delta)=(\delta_2^2\delta_3)^{(i_1,j_1)}(\delta_1\delta_3^2)^{(i_2,j_2)}$, if we set $i=\min(i_1, i_2)$, $j=\min(j_1, j_2)$,

$$\sum_{0\leqslant p\leqslant i, 0\leqslant q\leqslant j} a_{pq}\int[(\partial/\partial z_1)^{(i_1-p,j_1-q)}(\partial/\partial z_2)^{(i_2-p,j_2-q)}(\partial/\partial z_3)^{(p,q)}\varphi](z)\psi(z)dz.$$

As for $a_{pq}$ there are sixty-seven cases in total under the distinct conditions. For instance, if $\lambda_k, \mu_k$ are all positive integers and we take $i_1=i_2=\lambda_1+\lambda_2, j_1=j_2=\mu_1+\mu_2$, then $a_{pq}={}_iC_p\,{}_jC_q\ \lambda_2(\lambda_2-1)\cdots(\lambda_2-p+1)\ \mu_2(\mu_2-1)\cdots(\mu_2-q+1)$.

6. An intertwinning operator $A$ of $\mathcal{D}_\chi$ into $\mathcal{D}_{\chi'}$ is a continuous operator such that $T_g^\chi A=AT_g^{\chi'}$. From each invariant bilinear form we can obtain immediately the intertwinning operator by putting $B(\varphi, \psi)=\int(A\varphi)(z)\psi(z)dz$ for $\varphi\in\mathcal{D}_\chi$ and $\psi\in\mathcal{D}_{\chi'^{-1}}$. From the results in 5 we obtain the main theorem.

**Theorem.** *Among the representations $\{T^\chi, \mathcal{D}_\chi\}$ there exist following types of intertwinning operators* $(k=1, 2)$:

1) *identity operator;*

2) $$A_k\varphi(z)=\gamma(\lambda_k, \mu_k)\int z_k^{(-\lambda_k-1,-\mu_k-1)}\varphi(z_kz)dz_k,$$

*where*

$$\gamma(\lambda_k, \mu_k)=\frac{\Gamma((\lambda_k+\mu_k+|\lambda_k-\mu_k|+2)/2)}{\pi\Gamma((-\lambda_k-\mu_k+|\lambda_k-\mu_k|)/2)}2^{\lambda_k+\mu_k}\sqrt{-1}^{-|\lambda_k-\mu_k|};$$

$A_k$ *maps* $\mathcal{D}_\chi$ *into* $\mathcal{D}_{\chi^{s_k}}$; *in this case* $(\lambda_k, \mu_k)$ *can be both positive integers;*

3) *When* $\lambda_k$ *is a positive integer* ($\mu_k$ *any integer*),

$$A_k\varphi(z)=\int\delta^{(\lambda_k,0)}(z_k)\varphi(z_kz)\,dz_k;$$

$A_k$ *maps* $\mathcal{D}_\chi$ *into* $\mathcal{D}_{(\lambda^{s_k},\mu)}$;

4) *When* $\mu_k$ *is a positive integer* ($\lambda_k$ *any integer*),

$$A_k\varphi(z)=\int\delta^{(0,\mu_k)}(z_k)\varphi(z_kz)dz_k;$$

*A maps* $\mathcal{D}_\chi$ *into* $\mathcal{D}_{(\lambda,\mu^{s_k})}$.

*Every non-trivial intertwinning operator is expressed by a product of operators of the above types. For instance, for the operator A obtained from the invariant bilinear form of* (1) *in* 5, *we have* $A=A_1A_2A_1$ *or* $A_2A_1A_2$ *in the notation in* 2).

## References

[1] Gelfand-Graev-Vilenkin: Generalized function. V (in Russian). Moscow (1962).
[2] Gelfand-Neumark: Unitäre Darstellungen der klassischen Gruppen. Akd. Verlag, Berlin (1959).